Supplement: Appendix to Artsy Pseudo-Hamiltonian Tours

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In this Appendix prepared by the first author we first give an algebraic solution to the problem of the {6/(3,2)} and {6/(4,3) passing patterns, and then establish several theorems that characterize and count $\{n/(a_1, a_2, a_3,\ldots, a_m)\}$ designs and show their connection to a variation on Hamiltonian tours of cycles C_n, as explained in the primary Bridges paper.

{6/(3,2)} and {6/(4,3) passing patterns problem

We need to calculate separately for even and odd numbers of passes. If the total number of passes is x, then for ease of calculation, we will let y be the number of passes of the first weights in each sequence, 3 and 4. If x is even, then the number of weight 3 and weight 2 blue ball passes and the number of weight 4 and weight 3 red ball passes are each $y = \frac{x}{2}$. If the total number of passes x is odd, as in the solution shown in Figure 2(b) in the primary Bridges paper, then the number of weight 3 blue ball passes and the number of weight 4 red ball passes are each $y = \frac{x+1}{2}$, while the number of weight 2 blue ball passes and the number of weight 3 red ball passes are each $y - 1 = \frac{x-1}{2}$.

For even numbers of passes we solve $1 + 3y + 2y \equiv 2 + 4y + 3y \pmod 6$. This simplifies to $4y \equiv 1 \pmod 6$ which has no solutions since $2 = \gcd(4,6)$ is not a divisor of 1, and the balls never meet on an even pass. For odd numbers of passes we solve $1 + 3y + 2(y - 1) \equiv 2 + 4y + 3(y - 1) \pmod 6$. This simplifies to $4y \equiv 0 \pmod 6$, with solutions $y \equiv 0$ or $3 \pmod 6$ which can be represented by $y = 6k$ or $6k + 3$ for k any non-negative integer. Since $y = \frac{x+1}{2}$, solving for x gives $x = 2y - 1 = 12k - 1$ or $12k + 5$, shown in Table 2 in the primary Bridges paper by the shaded blue columns.

Characterizing and counting $\{n/(a_1, a_2, a_3,\ldots, a_m)\}$ designs

With respect to the n vertices of the cycle C_n let m be an integer in the set $\{1,2,3,\ldots,n\}$ and let $s_i \equiv a_1 + a_2 + a_3 + \ldots + a_i \pmod n$ for $i = 1,2,3,\ldots,m$. The *design* $\{n/A_m)\} = \{n/(a_1, a_2, a_3,\ldots, a_m)\}$ is the directed multigraph (typically) beginning at vertex 0 with edges successively of weight $a_1, a_2, a_3, \ldots, a_m$, continuing with another sequence of edges of weight $a_1, a_2, a_3, \ldots, a_m$ until an a_m edge first terminates at 0. This will first occur when the sum of the edge weights in the overall sequence reaches $\mathrm{lcm}(n, s_m)$, the least common multiple of n and s_m, forming a circuit through a subset of the vertices of C_n. If the design $\{n/A_m)\}$ includes each vertex of C_n exactly k times we say that it is also an *$(a_1, a_2, a_3,\ldots, a_m)$-step k-Hamiltonian tour of C_n*. If $k = 1$ and each vertex of C_n appears once then the design may be called an $(a_1, a_2, a_3,\ldots, a_m)$-step Hamiltonian tour of C_n. For convenience we define $S_m = (s_1, s_2, s_3,\ldots, s_m)$, and $d = \gcd(n,s_m)$. For $0 \leq k < d$ define e_k = the number of elements of S_m that are congruent to k, mod d, and let $E = (e_0, e_1, e_2,\ldots, e_{d-1})$. We will use E in the following discussion. See the examples in Figure A.1.

The edges of $\{n/(a_1, a_2, a_3,\ldots, a_m)\}$ are as follows, where it is convenient to reduce to elements of the set of least residues, mod n, $\{0,1,2,\ldots,n-1\}$, we have:

$(0, s_1), (s_1, s_2),\ldots,(s_{m-1}, s_m), \ldots,$
$(s_m, s_m+s_1), (s_m+s_1, s_m+s_2),\ldots,(s_m+s_{m-1}, 2s_m),\ldots,$
$\ldots,$
$\left(\left(\frac{n}{d}-1\right)s_m, \left(\frac{n}{d}-1\right)s_m + s_1\right), \left(\left(\frac{n}{d}-1\right)s_m + s_1, \left(\frac{n}{d}-1\right)s_m + s_2\right),\ldots, \left(\left(\frac{n}{d}-1\right)s_m + s_{m-1}, \left(\frac{n}{d}\right)s_m = \mathrm{lcm}(n,s_m)\right)$

Note that in a multigraph two vertices may be joined by more than one edge and some $\{n/A_m\}$ designs will include multiple edges rather than have the design traverse the same edge more than once.

Figures A.1 (a) and (b) show examples {12/(1,4,1,2)} and {8/(2,4)}. Note that in {12/(1,4,1,2)} the vertices 3, 7, and 11, which are each congruent to 3, mod 4, have degree 0; the vertices 0, 2, 4, 6, 8, and 10, which are congruent to 0 or 2, mod 4, have degree 2; and the vertices 1, 5, and 9, which are congruent to 1, mod 4, have degree 4. In {8/(2,4)}even vertices have degree four, odd vertices have degree zero, and multiple edges join pairs of even vertices. The Figure A.1(b) design duplicates that of a string loop tetrahedron held by four hands [4].

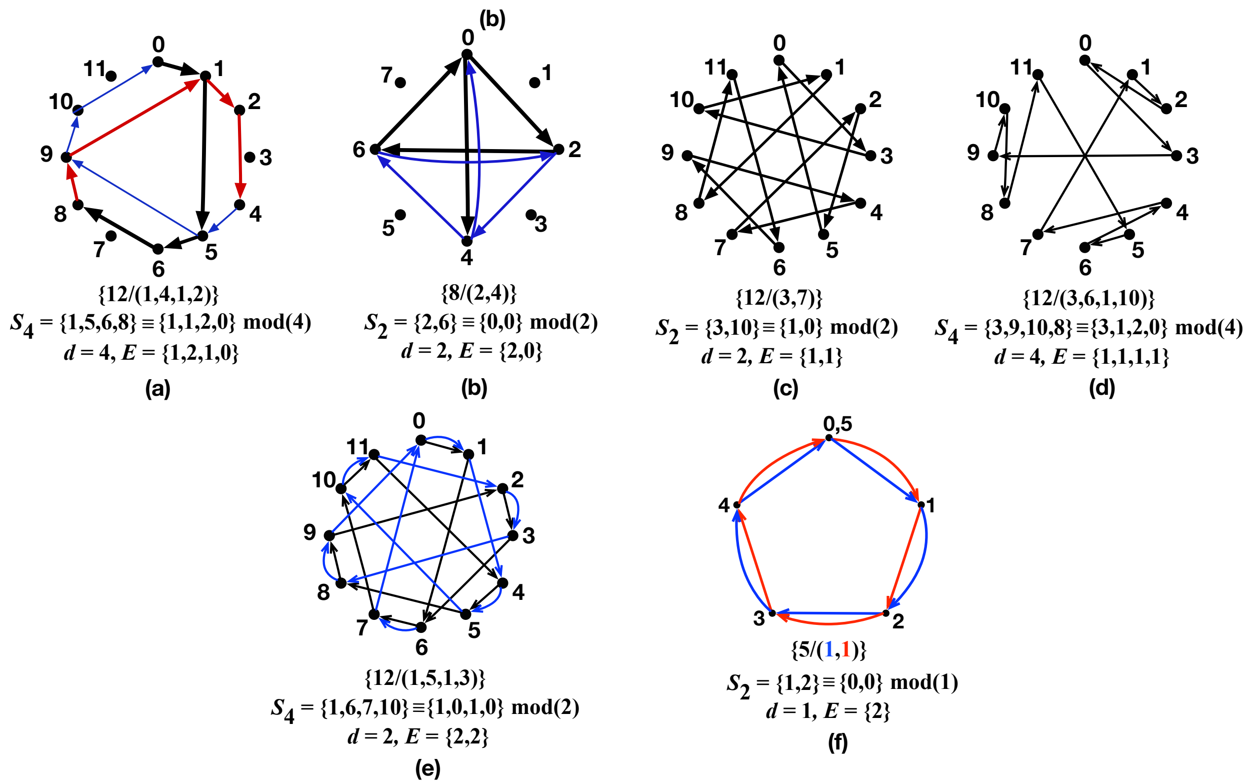

Figure A.1: *(a) {12/(1,4,1,2). (b) {8/(2,4). (c) {12/(3,7)}. (d) {12/(3,6,1,10)}. (e) {12/(1,5,1,3)}. (f) {5/(1,1)}.*

Since n is divisible by d the vertex labels of C_n, which are the elements {0, 1, 2, …, n–1} of Z/nZ, are naturally partitioned into equal size subsets congruent, mod d, to one of either 0, 1, 2, …, or d–1. For example, for {12/(1,4,1,2)}, $d = 4$ and those four subsets are {0,4,8}, {1,5,9}, {2,6,10}, and {3,7,11}. Since $S_4 = (1,5,6,8) \equiv (1,1,2,0)$, (mod 4), therefore $E = (1,2,1,0)$. For {8/(2,4)} we have $d = 2$, $S_2 = (2,6) \equiv (0,0)$, (mod 2), therefore and $E = (2,0)$.

We summarize parameters for $\{n/A_m\}$ in

Theorem A.1. The design $\{n/A_m\}$ on the vertices of C_n is a circuit with a total of $\frac{nm}{d}$ edges and in which $\frac{nm}{d}$ is also the number of times vertices appear in $\{n/A_m\}$. The degree of each vertex that is congruent to k, mod d, is $2e_k$. The total number of edges of weight a_i is $\frac{n}{d}$ times the number of times that value a_i appears in A_m.

Proof. The number of times the sequence A_m appears in the construction of$\{n/A_m\}$ is $\frac{n}{d}$ and each such occurrence of A_m gives rise to m edges so $\{n/A_m\}$ has a total of $\frac{nm}{d}$ edges. Each time that each of the m values a_i appears in A_m gives rise to $\frac{n}{d}$ edges of weight a_i in $\{n/A_m\}$.

We need to take care to understand whether vertices and edges are duplicated within the design or whether they appear uniquely. To calculate the degree of each vertex in $\{n/A_m\}$ we need to show that every value s_i of S_m generates exactly one pass of the circuit through each of the $\frac{n}{d}$ vertices of C_n that are congruent to s_i, mod d. The multiples of s_m, $\{s_m, 2s_m, 3s_m, \ldots, \left(\frac{n}{d}\right)s_m \equiv 0 \pmod{n}\}$, must be distinct, mod n, since if $xs_m \equiv ys_m \pmod{n}$ for $1 \le x < y \le \left(\frac{n}{d}\right)$ then $(y-x)s_m \equiv 0 \pmod{n}$ and $(y-x) < \left(\frac{n}{d}\right)$ contradicting the fact that $\left(\frac{n}{d}\right)s_m$ is the least common multiple of n and s_m. Since $d = \gcd(n,s_m)$, this set of $\left(\frac{n}{d}\right)$ multiples of s_m is identical to the set $\{d, 2d, 3d, \ldots, \left(\frac{n}{d}\right)d = n \equiv 0 \pmod{n}\}$ of $\left(\frac{n}{d}\right)$ distinct multiples of d, mod n. Similarly for any $1 \le j \le d$ and $0 \le x < y \le \left(\frac{n}{d}\right)$, we must have that $xs_m + j$ and $ys_m + j$ are distinct mod n. For any $1 \le i \le j \le d$ if $xs_m + i \equiv ys_m + j \pmod{n}$, then $(y-x)s_m + (j-i) = kn$ for some k. Reducing this equation, mod d, since s_m and n are both multiples of d, gives $(y-x)\cdot 0 + (j-i) \equiv k\cdot 0$, mod d, which would imply $i = j$, so $xs_m + i$ and $ys_m + j$ must be distinct. Each value s_i in S_m is of the form $xs_m + k$ for $0 \le k \le d-1$, as described above, and is congruent to k, mod d. In the $\left(\frac{n}{d}\right)$ occurrences of S_m in $\{n/A_m\}$ that s_i causes the circuit to pass through each vertex congruent to k, mod d, exactly once. Therefore, since e_k represents the number of times values s_i of S_m are congruent to k, mod d, e_k also represents the number of times $\{n/A_m\}$ passes through each vertex of C_n congruent to k, mod d. So any vertex congruent to k, mod d, will have degree $2e_k$ in $\{n/A_m\}$.

Corollary A.1.1. $\{n/A_m\}$ forms an A_m-step $\frac{m}{d}$ –Hamiltonian tour of the vertices of C_n if and only if all values of $E = (e_0, e_1, e_2, \ldots, e_{d-1})$ are identical.

Proof. By theorem 1, $\frac{nm}{d}$ is the total number of times the circuit passes through vertices. If all the values of $E = (e_0, e_1, e_2, \ldots, e_{d-1})$ are the same then also all vertex degrees will be the same, and the degree of each vertex will be $2\frac{1}{n}\frac{nm}{d} = \frac{2m}{d}$, and $\{n/A_m\}$ forms an A_m-step $\frac{m}{d}$ –Hamiltonian tour.

Suppose $\{n/A_m\}$ forms an A_m-step $\frac{m}{d}$ –Hamiltonian tour of the vertices of C_n. Then the degree of each vertex congruent to k, mod d, will be $\frac{2m}{d} = 2e_k$, so $e_k = \frac{m}{d}$ for all k since all vertices have the same degree in an A_m-step $\frac{m}{d}$ –Hamiltonian tour.

Example. Figure A.1(e) shows $\{12/(1,5,1,3)\}$ for which $d = 2$. Edges of weight 1 appear $2 \cdot \frac{12}{2} = 12$ times, and all vertices are degree $2 \cdot \frac{4}{2} = 4$. Figure A.1(f) shows $\{5/(1,1)\}$ for which $d = 1$. We may consider that every integer is congruent to 0, mod 1 since division by 1 leaves remainder 0 in all cases. All vertices are of degree $2 \cdot \frac{2}{1} = 4$. Since the edges alternate in color blue, red, blue, red,…, and $n = 5$ is odd, the design circles C_5 twice before the sequence of edges in $\{5/(1,1)\}$ ends with a red edge.

Corollary A.1.2 . The design $\{n/A_m\}$ is an A_m-step $\frac{m}{d}$ –Hamiltonian tour of the vertices of C_n if and only if m equals the number of times the tour passes through each vertex multiplied by the $\gcd(n,s_m)$.
Proof. This is just a restatement of the fact that $\frac{m}{d}$ equals the number of times the design passes through each vertex.

This allows us to easily specify examples of $\{n/A_m\}$ designs that are A_m-step Hamiltonian tours. For example, if $m = 1$ then we have the usual star polygon result that such a star polygon $\{n/k\}$ passes through each vertex of C_n if and only if $\gcd(n,k) = 1$. If $m = 2$ then we must also have $d = \gcd(n,s_m) = 2$. Since s_m must be a multiple of $d = 2$ the only possibility for S_m in this case is $S_2 \equiv (1,0)$, mod 2. This forces $A_2 \equiv$

(1,1), mod 2; in other words, the only designs $\{n/A_2\}$ that are A_2-step Hamiltonian tours of C_n are those in which a_1 and a_2 are odd and gcd(n, $s_2 = a_1 + a_2$) = 2. So if we pick two odd numbers, say 3 and 7 and a value of n which shares only the common factor of 2 with 3 + 7, say n = 12, then {12/(3,7)} forms a (3,7)-step Hamiltonian tour of C_{10}, see Figure A.1 (c).

If $\frac{m}{d} = 1$ then we must have $m = d$ so we have the following:

Corollary A.1.3.The design$\{n/A_m\}$ is an A_m-step Hamiltonian tour of C_n if and only if the following two conditions hold:
1. $d = \gcd(n, s_m = a_1 + a_2 + a_3 +,\ldots, + a_m) = m$.
2. The m sums $s_1 = a_1$, $s_2 = a_1+a_2$, , $s_3 = a_1+a_2+a_3$, …, , $s_m = a_1+a_2+a_3 +,\ldots, +a_m$, are distinct, mod m.

Given a value for m such as m = 6, we can use Corollary A.1.2 to show that the values of k such that there are A_m-step k-Hamiltonian tours of C_n are the divisors of 6, namely k = 1, 2, 3, and 6.

Corollary A.1.4. The number of values of k such that there are A_m-step k-Hamiltonian tours of C_n is $\tau(m)$ = the number of positive integer divisors of m.
Proof. $k = \frac{m}{d}$ so k must be a divisor of m for $\{n/A_m\}$ to be A_m-step k-Hamiltonian. We must also find an A_m and at least one value of n such that $\{n/A_m\}$ is an A_m-step k-Hamiltonian tour of C_n. Note that gcd($n = kd + d$, $m = kd$) = d, so let $n = (k + 1)d$. Let A_m = (1,1,1,…,1), a sequence of $m = kd$ ones. Then $s_1 = 1$, $s_2 = 2$, $s_3 = 3$,…, $s_m = m$ and each of the d values of $e_i = k$.

This tells us, for example, that a necessary condition for the existence of A_m-step 2-Hamiltonian tours of C_n is that m is even, that A_m-step 3-Hamiltonian tours exist only for m divisible by 3, etc. For example, Figure A.1(f) shows a (1,1)-step 2-Hamiltonian tour of C_5 in which m but not n is divisible by 2. In the example in the proof in which all edges are of weight 1 we might alternate edges of d colors.

Theorem A.2. Let n and $m \leq n$ be positive integers such that gcd(n,m) = m. Then there are (m–1)! distinct types of designs $\{n/A_m\}$ that are A_m-step Hamiltonian tours of C_n.

By "type" we mean that $s_1, s_2, s_3, \ldots, s_{m-1}$ are congruent, mod m, to a permutation of {1,2,3,..,m–1}, and s_m is congruent to 0, mod m. Actual values for the s_i may be chosen from {1,2,3,…,n}. Values for the a_i are then calculated from the s_i as described in the proof:

Proof. For m = 1 we simply have the 0! = 1 design type $\{n/k\}$ where gcd(n,k) = 1. For m > 1 there are (m–1)! sequences of partial sums of the form $S = (s_1, s_2, s_3,\ldots, s_{m-1}, s_m \equiv 0 \pmod d)$ where $(s_1, s_2, s_3,\ldots, s_{m-1})$ is one of the (m–1)! permutations of {1,2,3,..,m–1}. Each such set S generates an ordered m-tuple $A = (a_1, a_2, a_3,\ldots, a_{m-1}, a_m) = (s_1, s_2 - s_1, s_3 - s_2,\ldots, s_{m-1} - s_{m-2}, s_m - s_{m-1})$. That m-tuple A_m in turn generates the unique sequence of partial sums S. By Corollary A.1.3 $\{n/A_m\}$ is an A_m-step Hamiltonian tour of C_n.

For example, we will use these ideas to determine the number of A_m-step Hamiltonian tours of C_{12}. We first note that there are six possible values for $d = m = \gcd(12,s_m)$, namely the six divisors of 12: 1,2,3,4,6,12.

(1) $d = m = 1$. There are $\varphi(12) = 4$ positive integers 1,5,7, and 11 that are less than 12 and relatively prime to 12. Here φ is the Euler totient function where $\varphi(n)$ = the number of positive integers less than or equal to n that are relatively prime to n. Each gives rise to one A_1-step Hamiltonian tour of C_{12}, the four star polygons {12/1}, {12/5}, {12/7}, and {12/11}. We note that as undirected graphs {12/1} and {12/11} appear identical, as do {12/5} and {12/7}, though we will not denote those as the same since there may be applications in which the differences as directed graphs are important.

(2) $d = m = 2$. $\varphi\left(\frac{12}{2}\right) = 2$ since 1 and 5 are relatively prime to 6, and these give possible values of s_2 of $2\cdot1 = 2$ or $2\cdot5 = 10$ since they are the positive integers less than 12 that have gcd of 2 with 12. Then s_1 must be congruent to 1, mod 2, so its $\left(\frac{12}{2}\right) =$ six possible values are 1,3,5,7,9, and 11. So there are $(2–1)!\cdot2\cdot\left(\frac{12}{2}\right) = 12$ A_2-step Hamiltonian tours of C_{12}. These are {12/(1,1)}, {12/(1,9)}, {12/(3,11)}, {12/(3,7)}, {12/(5,9)}, {12/(5,5)}, {12/(7,7)}, {12/(7,3)}, {12/(9,5)}, {12/(9,1)}, {12/(11,3)}, and {12/(11,11)}. Note that there is significant duplication here, for example, {12/(5,5)} is identical to {12/5}, though in some applications we might want to alternate colors of the weight five edges. Also {12/(7,3)} and {12/(3,7)} will be mirror images. For now we avoid cataloging or counting types of duplication. See Figure A.1(c).

(3) $d = m = 3$. $\varphi\left(\frac{12}{3}\right) = 2$, giving $s_3 = 3\cdot1 = 3$ or $3\cdot3= 9$. There are 2! mod 3 choices for S_3, (1,2,0) or (2,1,0). For $S_3 = (1,2,0)$ there are $\left(\frac{12}{3}\right) = 4$ choices for values of s_1 that are congruent to 1, mod 3: 1,4,7, or 10. Similarly there are 4 choices for s_2 that are congruent to 2, mod 3: 2,5,8, or 11. Thus the total number of A_3-step Hamiltonian tours of C_{12} is $(3–1)!\cdot2\cdot\left(\frac{12}{3}\right)\cdot\left(\frac{12}{3}\right) = 64.$, for example {12/(7,5,9)}, {12/(7,5,3)}, {12/(1,8,9)}, etc.

(4) $d = m = 4$. $\varphi\left(\frac{12}{4}\right) = 2$, so $s_4 = 4\cdot1 = 4$ or $4\cdot2 = 8$, and the total number of A_4-step Hamiltonian tours of C_{12} is $(4–1)!\cdot\varphi\left(\frac{12}{4}\right)\cdot\left(\frac{12}{4}\right)^3 = 324$. See Figure A.1(d).

(5) $d = m = 6$. Using the same algorithm the number of A_6-step Hamiltonian tours of C_{12} is $(6–1)!\cdot\varphi\left(\frac{12}{6}\right)\cdot\left(\frac{12}{6}\right)^5 = 3840$.

(6) $d = m = 12$. The number of A_{12}-step Hamiltonian tours of C_{12} is $(12–1)!\cdot\varphi\left(\frac{12}{12}\right)\cdot\left(\frac{12}{12}\right)^{11} =$ 39,916,800. These are simply the 11! permutations of the eleven vertices other than 0 of C_{12}.

For small values of m we can now easily tabulate all types of $\{n/A_m\}$ designs that give A_m-step Hamiltonian tours for any n by calculating a_1 to a_{m-1} from the $(m-1)!$ permutations of (1,2,.3,…,m–1):

Corollary A.2.1. (i) The design $\{n/(a,b)\}$ is an (a,b)-step Hamiltonian tour of C_n if and only if $\gcd(n,a+b) = 2$ and a and b are odd.
(ii) The design $n/(a,b,c)\}$ is an (a,b,c)-step Hamiltonian tour of C_n if and only if $\gcd(n,a+b+c) = 3$ and either $a \equiv b \equiv c \equiv 1 \pmod 3$ or $a \equiv b \equiv c \equiv 2 \pmod 3$.
(iii) The design $\{n/(a,b,c,d)\}$ is an (a,b,c,d)-step Hamiltonian tour of C_n if and only if $\gcd(n,a+b+c+d) = 4$ and (a,b,c,d) is congruent, mod 4, to either (1,1,1,1), (3,3,3,3), (1,2,3,2), (2,3,2,1), (3,2,1,2), or (2,1,2,3).

The example above for C_{12} establishes the pattern for C_n, though we would want to pay attention to duplications or ignore less interesting examples such as the $(n-1)!$ permutations of $n-1$ of the vertices:

Theorem A.3. The number of A_m-step Hamiltonian tours of C_n is $\sum\left[\varphi\left(\frac{n}{m}\right)(m-1)!\left(\frac{n}{m}\right)^{m-1}\right]$, where the summation is taken over all factors m of n, and φ is the Euler totient function.

References
See the references in the primary Bridges paper.