Hans Hinterreiter's Flowing Fields

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Abstract

Hans Hinterreiter (1902-1989) was a Swiss painter who developed a formal geometric theory of "Concrete Art". He specified notations for his constructions and his colors that allow a concise characterization of any of his artworks. Considerable mathematical thought went into the design of his system, particularly his development of nonlinear geometric deformations that flowingly fill rectangles and circles. This paper explains some mathematical aspects of his methods and illustrates software I wrote to allow his designs to be reconstructed from his symbolic descriptions. Hinterreiter achieved an astounding synthesis of artistry and mathematical rigor that led to museum-worthy results. He deserves to be better known as a model of systematic artistic practice.

Introduction

I know of no one with so intense an obsession over mathematical art as Han Hinterreiter (HH). Born in Switzerland in 1902, he studied mathematics for one year at the University of Zurich then switched to the Swiss Federal Institute of Technology (E.T.H. Zurich) where he obtained an architecture degree in 1925, but only because his parents would not allow him to enter the painting profession. After three unhappy years of architectural practice, he largely removed himself from society and devoted his life to painting and printmaking. He painted for six years in "a lonely Alpine hut" above the remote Swiss mountain village of Seelisberg, then lived most of his life starting 1935 on the island of Ibiza, Spain. In his intellectual seclusion he was influenced by reading Wilhelm Ostwald's works on systematizing color and form [10] and developed his own unique conception of visual art as a highly structured mathematically based practice. Inspired by reading Einstein, he explored ways to warp the spaces of what he called his "color poems" or "eye music". In midlife, after forming his signature style, he befriended the artist Max Bill, who helped him exhibit and sell some work, but HH remained little known with no major exhibitions until over the age of 70. One year before his 1989 death he achieved a solo show at the Guggenheim museum in New York City, the pinnacle of his profoundly lonely career.

It is not easy to obtain information about HH. He published three books, two in German [5, 7] and one [6] a translation of the first into English. They are essential primary reading, but all are rare, expensive, limited editions. There are two very useful secondary sources, one in German and English [1], and one in Spanish [2]. There is no book-length biography, nor any published catalog of his complete works. To help guide the interested reader, I have prepared an annotated bibliography of my sources as a supplement to this paper [4].

I can not easily convey the extraordinary scope of HH's vision. He considered his work to be "a cultural achievement and task for future generations." This might sound boastful, but I am told by a native German speaker that the tone is instead joyful that he found a rational basis for art. His dense 800-page typewritten book *The Art of Pure Form* [7] is just the fundamental framework for understanding his techniques. While containing thousands of hand-drawn blueprint images, it does not include any artwork examples; it merely presents foundational material and grounding in his language of symbolic notations. He knew few could follow in his footsteps, realized that he had "exceptional mathematical talent," and stated "unfortunately, most of today's painters do not have the special talent necessary for the successful development of this new landscape." One requirement, he realized, is that "to be successful in this new art, one will have to learn this symbolic language, just as a musician must know musical notation." HH makes clear how alone he feels and how in his writing he is "reaching out a hand" to future artists.

I love many of HH's artworks. Wanting to better understand his thinking, I found it useful to replicate his compass-and-straightedge ideas in software. This allows one to easily mix and match his elements and create many new "hinter-Hinterreiters", but original art is not my goal here and software is not the topic of this paper. I see each opus as a puzzle and my software merely confirms if I have properly solved its geometry. (HH had some awareness of the possibilities of computer animation and near the end of his life suitable technology was available, but he never connected with appropriate experts.) His magnum opus gives guidelines for design and suggestions about complexity, noting, e.g., that his followers will need "fine psychological understanding and empathy" when making the many choices within his system, but apparently there are no followers. So I hope this exposition may at least increase the number of people who understand and appreciate his work.

Forms, Fields, and Colorings

HH systematized three aspects of his work: *forms*, *fields*, and *colorings*, summarized here and expanded upon in the examples below. They formally characterize the design of any of his paintings. He often recorded these specifications for a work in its margin, with hand-written notes in his custom notation.

A *form* is a set of line segments in a triangle, square, or hexagon. The shape's edge is divided by a small integer (typically $4 \le n \le 6$) to produce a grid, then segments are chosen that connect grid points. This results in a module that is repeated in a transformed manner across the canvas. Figure 1 (right) shows an early study, "ME 197", where the form (Figure 1, left) is triangular and its edge length has been divided by five to produce a grid of 25 small triangles. The legal grid points are the vertices and centers of these small triangles. HH invented a notation that allows the designer to specify the symmetry of the form and a set of segments. In Figure 1 the form has 3-fold rotational symmetry. Eight segments are specified, which are rotated to produce the 24 visible segments. HH created a large catalog of grids and segments drawn on tracing paper, so he could quickly overlay selected sheets to see what a form looks like. He did the combinatorics to calculate that the system admits "millions of wonderful designs". I won't explain HH's segment notation here except to mention that to efficiently encode rotational symmetry, he starts his indexing with point #1 at the center of the grid and works outward using a combination of digits and letters that concisely specify a set of symmetrically related segments [7].

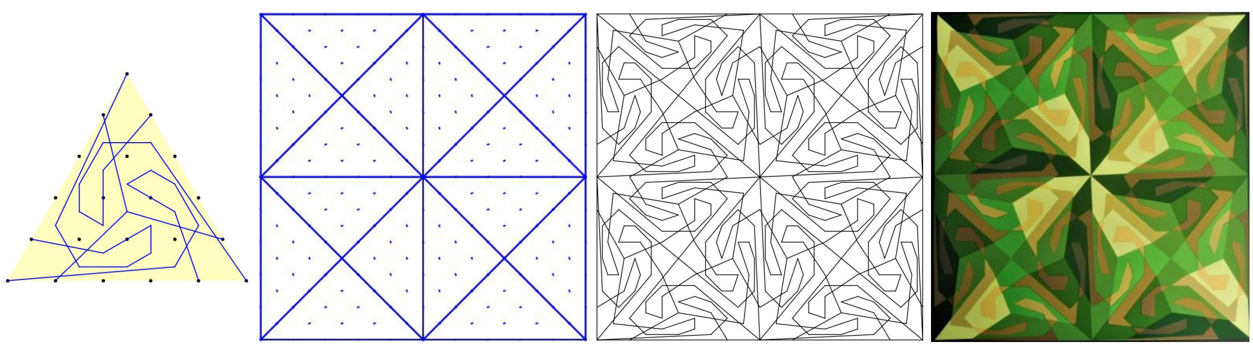

Figure 1: *Form, field, reconstruction, and original of Hinterreiter's ME197*

A *field* is a plan for laying out multiple copies of a form within a rectangle, circle, or other bounding frame. HH generally describes a field via text, formulas, and/or a drawing of its region boundaries. I implement fields with a set of deformation rules, which specify not just the boundaries but what exactly happens within the regions. In Figure 1, the field is implicit in the set of 16 isosceles right triangles and the evenly spaced dots. The equilateral form is to be mapped to each isosceles region with a simple linear transformation, but HH more often used nonlinear mappings, as discussed below. This example is also simple in that the sixteen triangles are congruent, while in general each region within a field can involve a different warping. HH's fields are based on triangular, square, or hexagonal regions. They generally pack their frame nicely in the sense that he tries not to truncate small bits of a form that would be mapped outside the picture frame. (But cutting a form exactly in half is allowed, as seen below.) Designing interesting nonlinear fields appears to have fascinated HH, as over 500 pages of [7] are devoted to fields and their interrelationships, with over 2000 illustrations.

HH's *colorings* are based on Wilhelm Ostwalds's color space [10], which provides specific symbols for a discrete set of colors and provides guidelines for placing different colors together in a composition. (The system consists of four primary colors, red, blue, yellow, and green, plus twenty secondary colors, all placed in a circle around the circumference of a dicone that blends in discrete steps to black at one apex and white at the other.) HH would mark his designs with the symbols for the colors he planned to use, then would fill in the regions like a color-by-number painting. Typically there are more colors than distinct shapes in a painting and the areas are filled using a sub-symmetry of the outline's geometric symmetry [8]. (For example, in Figure 1, darker and lighter hues occupy regions of the same shape.) HH often executed the same design multiple times with different color experiments. Some 92 pages of [7] are devoted to coloring, but the subject is beyond the scope of this paper. Filling specified colors into bounded regions of an image is straightforward, so I am content here just to show line-drawing reconstructions of forms mapped to fields.

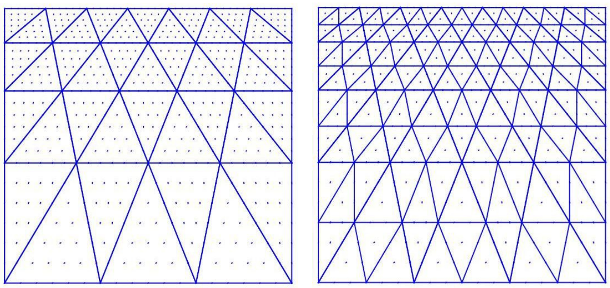

Figure 2: *Field with linear divisions; subdivided.*

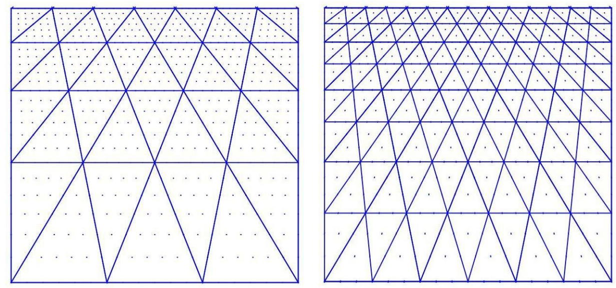

Figure 3: *Field based on perspective grid; subdivided.*

Several examples of HH's fields are analyzed below, with an emphasis on his mathematical thinking, but first it is useful to review some elementary mappings for comparison. Image warping is commonly accomplished with a linear map from triangles to triangles with each point transferred by using its barycentric coordinates [3] (as I did sixteen times in my reconstruction in Figure 1). But independent linear maps are unlikely to connect segments smoothly at the region boundaries. Figure 2 and 3 each show a field with the same 40 triangular region boundaries. In both cases, the region boundaries derive from the same two off-image "vanishing points" (towards the back left and back right) and the dots indicate how the space is warped within each triangle, but the dot positions are subtly different. Figure 2 uses a linear mapping, which is simple to implement, but produces a kink (a slope discontinuity) when we draw smaller triangles along the dots. A perspective mapping is used in Figure 3, resulting in smooth connections. Although HH never explicitly explained this issue (now called "C_1 continuity" [3]), it is clear that he sought out fields that eliminate these kinks, even if such fields are more complex to implement.

Another simple but unsuitable method of image warping is the standard "bilinear" mapping to quadrilaterals ("quads"), shown in Figure 4 [3]. It works linearly on the four edges of a square and maps horizontal or vertical segments of the form to straight segments in the field, but slanted segments in the form are mapped to parabolas, e.g., the red diagonals. The segments and parabolas will not be suitable for the final image if they do not connect smoothly to the adjacent regions of the field. As with the triangle mapping of Figure 2, this method allows each region to be mapped independently, but that is actually a weakness as it does not impose any wider regularity and slope continuity at the boundaries within the field.

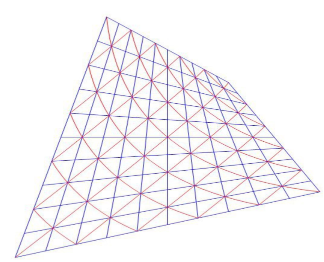

Figure 4: *Bilinear mapping*

The painting in Figure 1, titled ME 197, is an early study without opus number. Its field lacks two properties HH later adhered to. In his later work, the regions are not mapped independently (as in Figure 2), but instead there is a coherent flow across the image (as in Figure 3). Secondly, he avoided fields (as in Figure 1) that are based on a single shape arranged in a tessellation that could be extended indefinitely. HH had visited the Alhambra in 1934 and was greatly impressed with the tessellations he saw, but he felt that they are too "crystalline" and do not make a finished work of art, because they can be extended indefinitely or terminated arbitrarily, so lack inherent closure. He developed ways to make "a completely self-contained picture, where a continuation outside the frame is neither conceivable nor sought nor suspected". Although he produced many sketches in which forms repeat in regular tessellations of triangles, squares, or hexagons, these are merely preparatory drawings or color studies. In the remainder of this paper, six of HH's smoothly varying fields from his mature artworks are briefly examined.

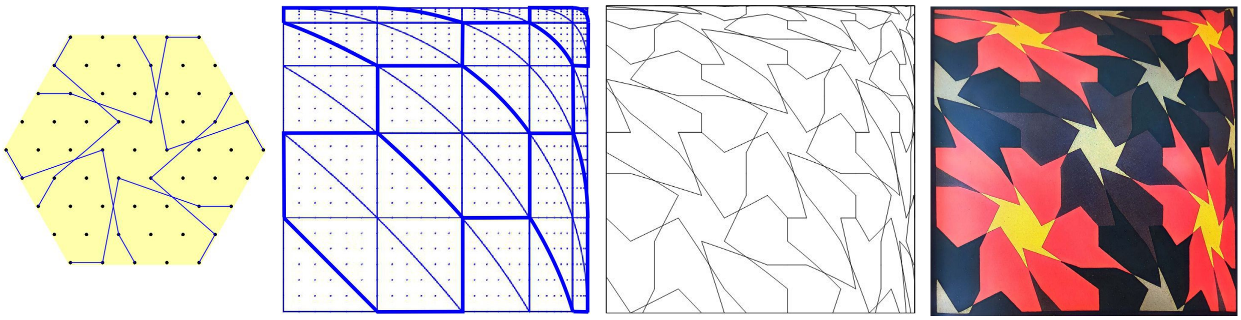

Figure 5: *Form, field, reconstruction, and original of Hinterreiter's Opus 115B*

Example 1. Opus 115B

Figure 5 reconstructs HH's Opus 115B, which is based on a hexagonal form that is compressed horizontally and vertically in a field derived from a sine function. Its continuous nature is in line with HH's idea that a field should "operate like a magnetic field". To reconstruct it, first draw six vertical lines, not evenly spaced, but instead at a position of $x = \sin(90\ i/5)$ as i goes 0, 1, 2, 3, 4, 5. Horizontal lines are similarly spaced to create a 5×5 grid of rectangles. Slice each rectangle on its "diagonal" to make two curve-edged triangles. One can see how these slices should be curved by following the dots that indicate how the sine distortion works within each rectangle. Then the 50 triangles can be grouped by sixes into hexagons (with some partial hexagons undesirably present at the frame edges). This may seem like a complex construction, but it is one of the relatively simple examples in the book. HH gives a straightedge, protractor, and compass method for this family of field, allowing for an arbitrary number of main divisions horizontally and vertically, not necessarily 5, plus one might choose a tangent function instead of the sine function. Mixing and matching such options and the direction of the slice, etc., gives many possibilities. It seems HH was very comfortable drawing complex constructions from his mathematics and architecture training and expected his followers to patiently fill in many details.

The form in Figure 5 is hexagonal with a grid of order 4. There are four basic segments, each repeated with six-fold symmetry around the hexagon. When the 24 segments of this form are mapped into the field hexagons (and partial hexagons along the borders) the result is the design of Figure 5. HH's handwritten construction notes for this painting are shown in Figure 6. The form *4*/6 98a+82a+52e+75a* is HH's notation for the four basic segments in a hexagon with 6-fold rotational symmetry (and not mirror symmetry). The *Sinus funktion Feld* line gives the specific parameters for the sine-based field. The *Farbe* diagram specifies the Ostwald colors used to fill in the regions and how they relate to each other. If one understands his constructive system, these three items are enough information to reconstruct his design (except that as far as I understand it, the notation doesn't specify precisely where in the outline each color goes).

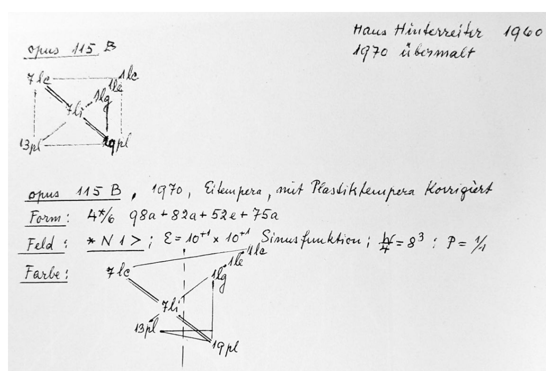

Figure 6: *Hinterreiter's marginal notes specifying in his notation the form, field and coloring of Opus 115B*

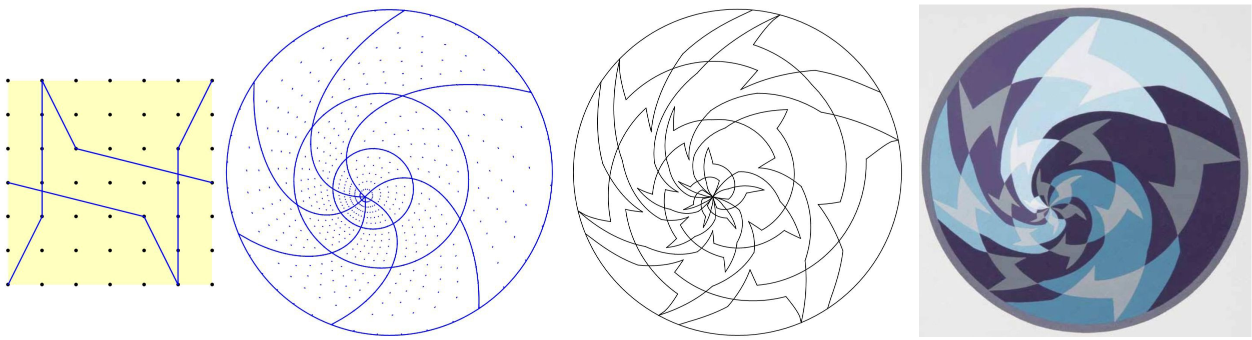

Figure 7: *Form, field, reconstruction, and original of Hinterreiter's Opus 80B*

Example 2. Opus 80B

Opus 80B, shown in Figure 7, is one example of how HH mapped forms into a circular frame. The form is square, based on a 6×6 grid. The field is based on nested circles, each divided into six curved quadrilaterals. Offset near the center, the mapping has a singularity where one edge of the six inner quads collapses to a point. Many of HH's fields involve a singularity of some sort as a focus of visual interest. For this family of fields, HH outlines options for the number of enclosing layers, the number of divisions around the circumference, the position of the center, the rate at which the circles shrink, the shape of the radial lines, etc. My reconstruction is based on estimated parameters, close to the ones he must have used.

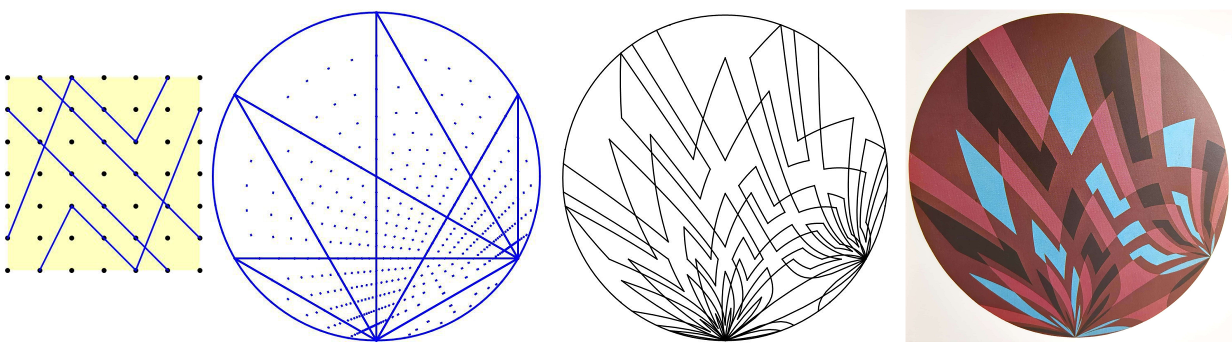

Figure 8: *Form, field, reconstruction, and original of Hinterreiter's Opus 84*

Example 3. Opus 84

In Opus 84, analysed in Figure 8, HH presents a brilliantly different solution to the challenge of mapping square forms into a circular frame. Here the field has two singularities, both on the circumference. There are two equiangular fans of rays that emanate from the singularities and cross to outline a pattern of quadrilateral regions. The quads that touch the singularities degenerate to triangles since one edge of each is collapsed to a point. The regions that would cross the circumference have been sliced on a "diagonal" into a curved triangle. What appears to be one segment connecting the singularities is actually two co-linear quadrilateral edges. To understand why pairs of rays meet exactly at the circumference, recall the Euclidean theorem that the angle between two points on the circumference as seen from another point on the circumference is half the angle as seen from the center.

(The curves in this field are of an interesting class that is not well known. Imagine the second hands of two watches, one placed on each singularity. They rotate at the same constant velocity but might not be parallel. Their intersection draws a circle because the angle between the moving lines is constant. This is the special 1-to-1-ratio case of the question of what curve results when an arbitrary segment from a form is mapped to this field. Each possible slope of a segment in the form corresponds to a ratio between the speeds of the two watch hands. It is a nice application of bi-angular coordinates [9] and the general answer is a sectrix of Maclaurin, with each possible segment slope giving a special case [11].)

Example 4. Opus 23

The most interesting of HH's fields are based on the properties of tangents to conic sections and involve intricate constructions. Figure 9 is one of many blueprint drawings in [7] and gives a sense of the book's architectural aesthetic. This field is made by first drawing the upper half of an ellipse. Then an equi-angular fan of rays ($n = 16$ rays in this example) is drawn centered at one focus of the ellipse. The eccentricity of the ellipse was chosen so that two of the rays (here, the rays numbered 5) pass through the ends of the semi-minor axis. Then at each intersection of a ray and the ellipse one draws the tangent line. The field is bounded by the rectangle ABEF that lies above the ellipse. The left and right sides are parallel because they are the tangents based on the ends of the semi-minor axis. The bottom of the rectangle is orthogonal to them because it is the tangent at one end of the semi-major axis. The lovely feature of this construction is that pairs of tangents meet at points along a straight line across the top (line BCE), even for higher values of n. As HH points out, this is because that line is the "polar" of the focus in the conic, which is a kind of reciprocal defined in projective geometry. HH achieves a rectangular field with a clean boundary through deep and subtle principles. In discussing the general case, HH explains an efficient way to draw the tangent lines and how only certain combinations of n and the eccentricity are compatible.

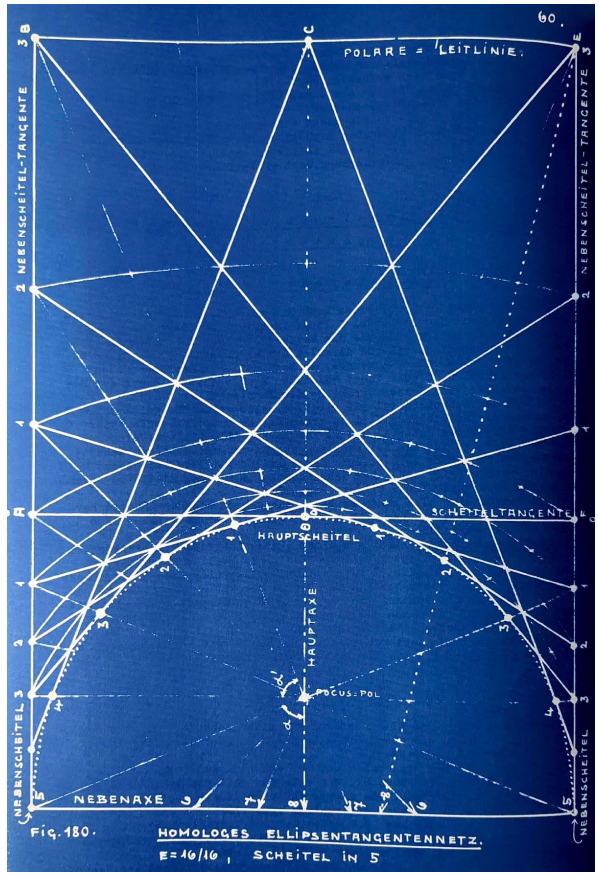

Figure 9: *Construction of a field based on tangents to an ellipse. (Figure 180 of [7])*

Opus 23 is based on a simpler instance of this general construction, but using $n = 10$ rays and with the rays numbered 3 hitting the ends of the semi-minor axis. Figure 10 shows its form is based on a 6×6 square grid and it is mapped to eight and a half quadrilateral regions in the field. The top quad is sliced in half on a diagonal by the polar line. The region at the bottom center looks like an isosceles triangle but is in fact a quadrilateral with two sides co-linear, meeting at a vertex at the end of the semi-major axis.

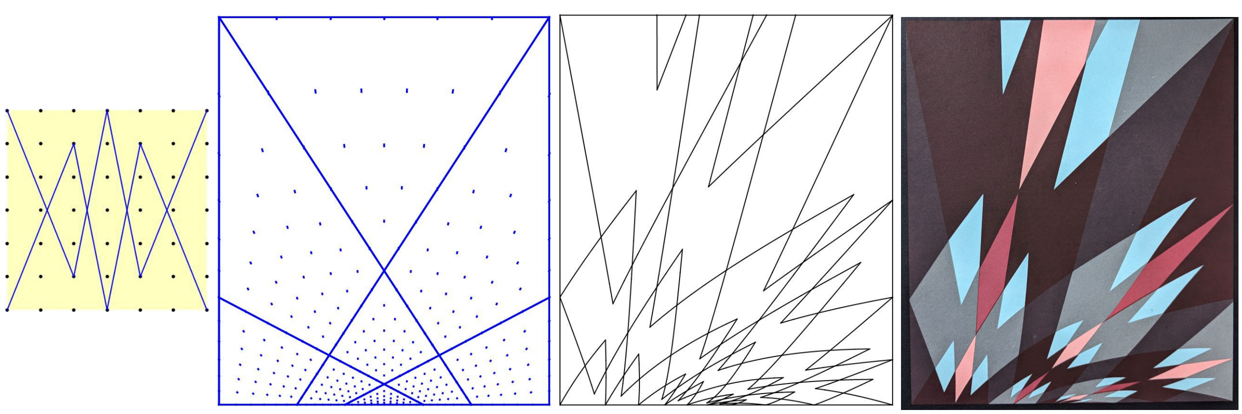

Figure 10: *Form, field, reconstruction, and original of Hinterreiter's Opus 23*

It is fascinating how both the form and the field are mirror symmetrical, but their mirror directions do not align, so the result lacks symmetry. Opus 23 is one of only two artworks that HH specifically discusses in his books: "Though unsymmetrical, the picture contains the two symmetries hidden in itself, which gives it a special charm. It shows that by means of this new language of signs for abstract forms you can easily explain things which formerly could neither be taught nor explained." [5, 6] It is not clear to me what exactly he means here, but I quote this as it is one of very few statements I have found where HH discusses a specific work of his. Oddly, this statement about Opus 23 appears in [6] without any image of Opus 23, so the reader of that work has little chance of understanding the construction or his meaning. (Note: It is possible that I have the image upside-down, but I am following the orientation of HH's presentation of the field in [5] and [6].)

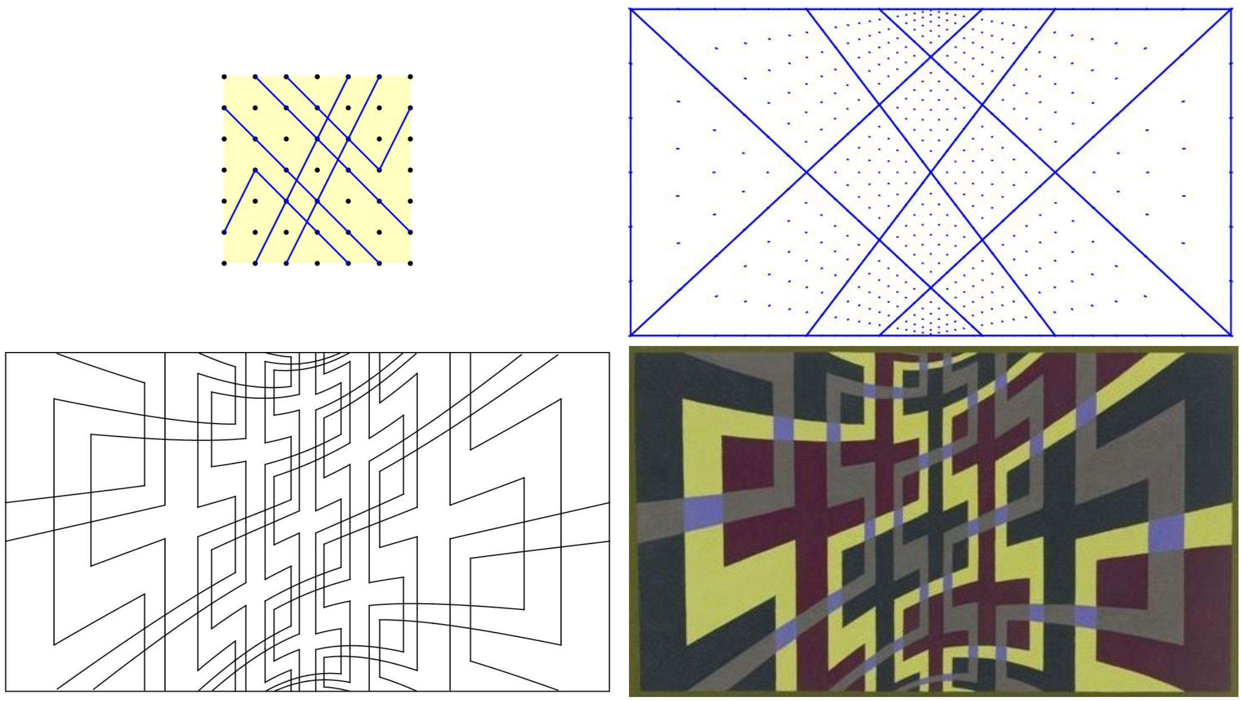

Figure 11: *Form, field, reconstruction, and original of Hinterreiter's Opus 83*

Example 5. Opus 83

Figure 11 shows the structure of HH's Opus 83. The form is again based on a 6×6 square grid. It is interesting to compare it to the similar form in the subsequent Opus 84 (Figure 8) and to note that the differences in field and coloring completely hide the similarity in form. The field here is analogous to Opus 23, but instead of an ellipse it derives from tangents to the two branches of a hyperbola. Again, HH provides general instructions and a detailed illustration for drawing the field with straightedge, protractor, and compass [7]. (The two lines crossing at the center of the field are the asymptotes to the hyperbola, where a sweep of tangents switches from one branch to the other.) As the dots indicate, the left and right triangles each get mapped with only half of the form, but the small triangles at the top and bottom center of the field are actually full quadrilaterals with two edges co-linear. It is enormously complex to construct, but the final image has a wonderful dynamic energy.

Example 6. Opus 43

I will briefly mention Opus 43 because it is the only other artwork that HH specifically discusses [5, 6] and because again there is no image of the work in [6] to help the reader. Its field is based on tangents to an ellipse plus rays from one focus. As shown in Figure 12, here n=16 equiangular rays emanate from the focus of an ellipse and the one numbered 6 is aimed at the end of the semi-minor axis. The quad regions of the field are bounded by tangents (green) to the ellipse at the intersection points and by the rays (blue) from the focus. The picture frame (gray) is bounded on the left by the vertical tangent, on the bottom by the x axis, on the top by the uppermost tangent, and on the right by the polar line to the focus. Then the quads are sliced on (curved) diagonals to make a field of triangles. The left edge of this frame does not align with the region boundaries and six of the triangles protrude left of the frame. But the dots show how the six are sliced exactly in half and HH seems to feel this is compatible with his desire for ‟pictures which are complete in themselves″ and ‟provide an orderly limit″.

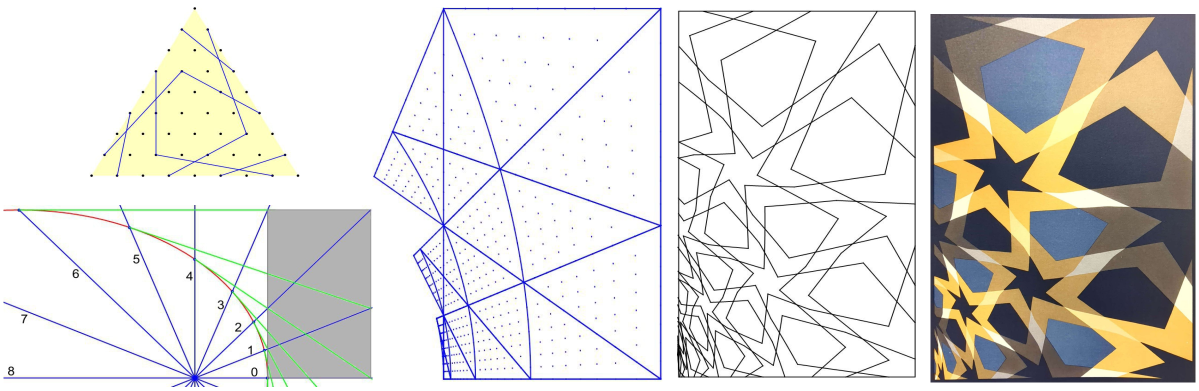

Figure 12: *Form, construction, field, reconstruction, and original of Hinterreiter's Opus 43*

Conclusion

I will give HH [7] the final words about his art, his big book, his fields, and where one may go from here:

> A work of art comes into being through the potency of three worlds of figurations: the crystalline elements which are incorporated into an appropriate field. (This alone gives millions of compositional possibilities). This wealth then has added to it the power of color, a practically inexhaustible realm...
>
> Like all works of a fundamental nature, this one will have to wait and will not be understood by middle-ranking minds, but rather attacked or even killed. But the author is not concerned with quick success; rather, he wishes for those rare readers of the highest intelligence and creative talent, whose view suddenly opens into a new wonderland of beauty...
>
> An orderly logical image limitation, the law of cessation within the infinite surface, must be part of the structure of the network. You have to have the feeling of finishing.
>
> Everyone can penetrate this new territory in their own way.

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