Double Take: Geometry, Perspective, and Optical Illusions

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Abstract

Perspective geometry, so it's said, was the art that made paintings seem realistic and true to life. Yet the same artistic techniques—and by extension, the same geometry—can create images that astound and confound us. We provide a carnival of such examples, from special effects in the movies, to sculptures that seem to move as we move, to hidden objects, and to objects whose mirror images transform them into other beings.

Optical Illusions that Use Perspective

Optical illusions are one of those lovely gifts that perspective geometry gives the active viewer. This notion might seem counter-intuitive: in the oversimplified way than many of us learned about the subject, the use of perspective transformed art from seemingly flat distortions to seemingly realistic depictions of the world. But while the magic of seeing a painting or a film that invites us into a larger, multi-dimensional space is indeed awe inspiring, there is more to perspective than realism.

In this paper we highlight several aspects of perspective that allow for compelling illusions. We particularly focus on the aspects that arise not just from the elements of the drawing or photograph itself—not just the vanishing points, the converging lines, the trapezoidal checkered floors—but from viewers' interaction with the canvas. In these optical illusions, we put the "looking" back into the analysis of the art.

We will discuss three such aspects of perspective illusions. Firstly, we will linger the longest on a variety of illusions that arise from choices of viewing target and distance or from ambiguity in those choices. From there, we look at illusions that use mirrors to create ambiguous or even false interpretations, and finally we touch on illusions that use non-planar canvases that appear to move as viewers change position.

Viewing Distance and Viewing Target

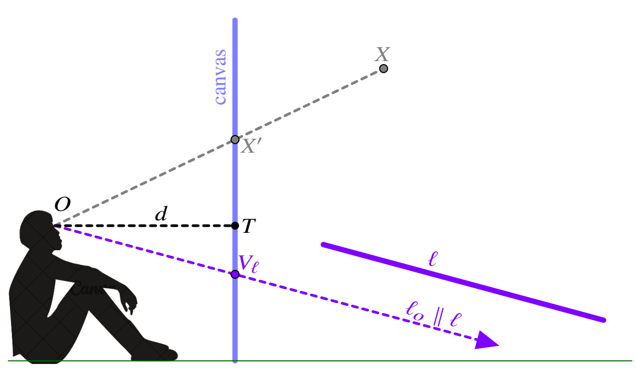
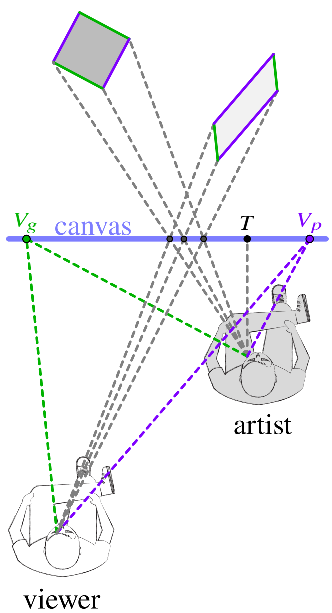


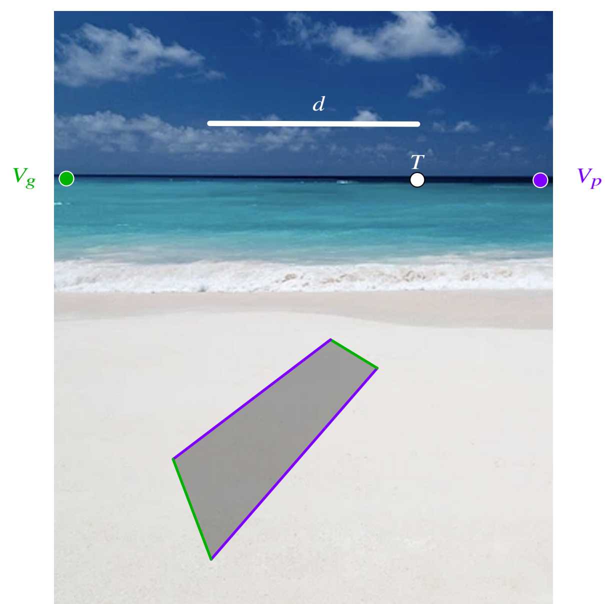
Figure 1: *A perspective set up, showing the observer O and canvas , with viewing target T, viewing distance d, as well as a point X with its image X' and a line ℓ with its vanishing point V_ℓ.*

In a typical perspective set-up (see Figure 1), we assume a planar canvas and an artist whose eye is considered to be a point O not on the canvas (sometimes called the *viewpoint*). In any geometrically correct perspective drawing or photograph on a planar canvas, there is a unique *viewing target* T (the point on the canvas closest to O) and a unique viewing distance d (the distance between the observer O and the target T). A point $X \in \mathbb{R}^3$ in the "real world" has an image X' located at the intersection of the artist's line of sight OX with the canvas. For a line $\ell \subset \mathbb{R}^3$ not parallel to the canvas, its vanishing point V_ℓ is the point on the canvas such that the line of sight OV_ℓ is parallel to ℓ (that is, if the artist is looking at the vanishing point, their line of sight is parallel to and never intersects ℓ). By this set up [5], lines that are parallel to the canvas have no vanishing points (that is, the images are parallel to the lines they portray); lines that are parallel to each other and not parallel to the canvas have images with a common vanishing point.

But beyond the location of the vanishing points themselves is the implication for both (a) where the artist was standing relative to the canvas and (b) the corresponding effect on other viewers. If a viewer stands in a location that is different from O, the picture may still seem to consist of images representing parallel lines in the real world, but their directions will appear distorted, because the direction from the viewer to V_ℓ is no longer parallel to ℓ. As directions appear distorted, so shapes also seem to warp.



(a)



(b)

Figure 2: *(a) a top view and (b) a canvas that shows the image of a square when viewed at a distance d from the viewing target T, and which appears to be a shape with odd angles from more distant locations.*

Figure 2 illustrates the effect of viewing location. In Figure 2(a), a top view shows an artist whose lines of sight to the two vanishing points V_g and V_p meet at right angles, as do the green and purple lines in the square in front of the artist. For viewers standing further away from the canvas, however, these lines of sight meet at acute angles, and so the drawing appears to be of an odd shape, with angles far from 90°. Figure 2(b) shows the corresponding drawing, in which the gray quadrangle appears stretched and misshapen to us. Any reader of this paper would have to put their eye essentially next to the book or screen to see the image "correctly" as a square: the viewer's eye would need to be at a distance of d directly in front of the viewing target T, almost a physical impossibility unless we enlarged the image (and hence d) significantly.

To be more specific, the quadrangle in Figure 2 looks distorted for both extrinsic and intrinsic aspects: both because we as viewers are far from T and also because the corners of the quadrangle within the picture are far from T. Artists know to keep the vanishing points far apart (that is, to keep the viewing distance large) in order to “avoid distortion”. A corollary to this piece of advice is to draw things close to the viewing target (here, “close” means compared to d) to avoid distortion, or far from the viewing target to force the viewer into a particular location.

A comparison of Trompe L’Oeil and anamorphism, as in Figure 3, illustrates how both intrinsic and extrinsic geometry can affect illusion. *Trompe L’Oeil* artwork tends to have large viewing distances appropriate to the physical space in which they appear, and so their illusions are easily accessible to viewers. Anamorphic art has a much smaller viewing distance, and thus requires that viewers look at the canvas “from an angle”—that is, close to the canvas at a point distant from the image. From this angle, however, anamorphic art can seem just as surprisingly realistic as Trompe L’Oeil, as fans of Hans Holbein’s *Ambassadors* or Julian Beever’s virally famous sidewalk chalk art [2] can attest.

Figure 3: *(left) Trompe l’oeil still life of a half-open wall cabinet by Franciscus Gijsbrechts [6]; (right) Anamorphose réalisée by Dimitri Parant [14].*

Viewing distance effects can lead to illusions in many ways. One familiar effect arises when we share our photographs of majestic panoramic views. Who among us has not had the experience of saying, “It was so much better in person; this picture doesn’t capture it at all!”? Neither the camera nor the photographer is at “fault” here; rather, viewing distance of a photograph is generally on the order of the size of the picture (depending on the camera, of course), which is generally much smaller than the distance from the arm’s length from which we normally look phones or photos. Viewers can recapture the sensation of “being there” either by making the photograph much larger or by viewing a small photograph with one eye from a much closer distance—often, uncomfortably close. Similarly, large, mural-sized perspective paintings lose much of their sense of depth when reproduced as small figures in an art history book.

Zoom/telephoto lenses in cameras take advantage of viewing distance effects. They allow us not only to make distant objects feel closer, but also to alter viewers’ relationships with an image, sometimes to dramatic effect. The “Hitchcock Zoom” (which goes by many other names as well) allows cinematographers to make the world seem to explode around a character. In this technique, first used in Hitchcock’s 1958 classic *Vertigo*, the camera simultaneously zooms in on character’s face while traveling backwards on a track. The effect is that the main character remains in focus and a constant size, but the scenery behind the character appears to swell in size, giving the audience a feeling of dread and anxiety that they can’t quite explain.

A more homegrown example of the Hitchcock zoom is accessible to anyone with a wheeled office chair and a hallway with a window at one end, especially if that window affords the view of buildings or trees. A person who sits on that chair and is wheeled backwards will get the odd sense that the world outside is getting closer, not further away. This arises because the objects outside the window will have relatively larger images compared with the window frame. Our eyes naturally zoom in; in actuality, the objects outside are not getting larger, but the window frame is getting smaller faster than the more distant objects do. One popular YouTube channel promotes this effect as the *Sydney Opera House Illusion* [4].

Perhaps the most intentional use of viewer location in creating illusions is the genre of "forced perspective", which takes many forms: in trick photography, a person might appear to have caught a giant fish or to be growing trees out of her head; the Ames Room appears to make its inhabitants shrink or grow as they walk from one wall to the next. The Ames Room pictured in Figure 4 (a,b) is trapezoidal, with a sloping floor and the left corner further from the viewer than the right corner is, but appears rectangular from the peephole where these photographs were taken.

(a) **(b)** **(c)** **(d)**

Figure 4: *(a, b) The author in two different corners of an Ames room; (c) an installation that can appear to be a chair from the right viewpoint but is actually made of (d) disconnected components.*

Forced perspective in environmental art can make object seem to appear out of disarray or disappear into the background, depending on the artist's intention. Jan Beutener's 1975 installation *The room* [11, plates 121–125] consists of a seemingly random collection of scraps of wood and other objects, but when viewed from a designated spot, appears to be a chair with a coat draped on it, a wooden box, and a step stool. The Museum of Illusions in Minneapolis has a similar display, shown in Figure 4 (c, d), delighting modern audiences. John Pfahl's photography [15] includes objects or markings placed in natural environments that create geometric shapes (squares, triangles) in the final photograph; it appears that the photograph itself has been marked up, but in reality, it is the landscape itself. Howard Lee, another environmental artist, reverses Pfahl's approach and paints portions of the landscape to make it invisible, so that a tree ([10] for example) appears to be split in the middle, missing its midsection through which the background seems to be visible. If viewers are forced into the proper location, the resulting illusion can be quite convincing, as the painted door in the author's "Kitchen of Many Delights" demonstrates (Figure 5).

Figure 5: *A door to the basement painted to blend with the window and wall behind it, in the author's kitchen. (Left) the door is open against the back wall, (middle) the door is ajar, (right) the door is closed and not visible in the photograph.*

In contrast to forced perspective, ambiguous figures such as those in Figure 6 admit multiple interpretations of the image, which correspond to differing interpretations of where the perceived object is relative to the viewer. The ball and the bars of the two cages in Figure 6 are identical except (to better illustrate the illusion) for which lines overlap each other or the ball. In the left-most image, the red, orange, and green bars appear to be closest to us, and we as viewers seem to be below the box; in the center image, the blue and purple bars appear closer, and we as viewers are above the box. On the right, as we switch our viewpoint (or even turn the page upside down), we can shift between seeing a large portion of chocolate cake with one piece missing, or a single piece of raspberry-frosted cake remaining on a largely empty platter.

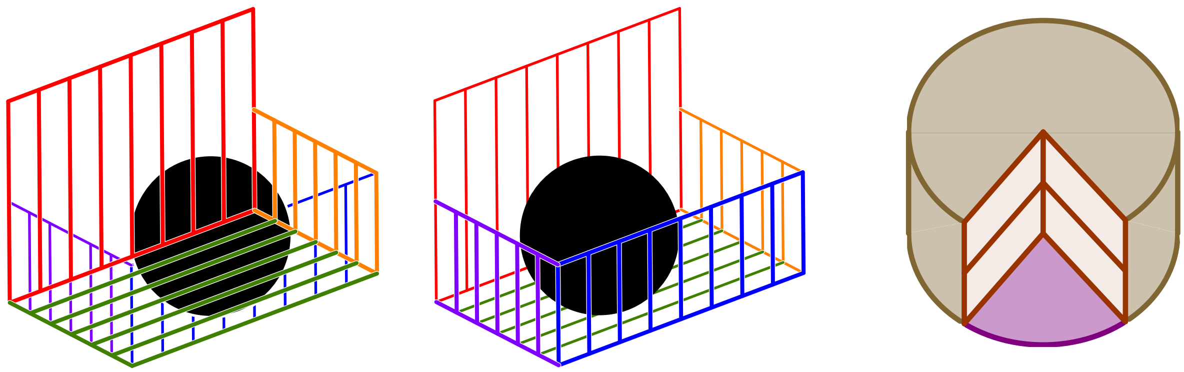


Figure 6: *Two views of an ambiguous cage, and an ambiguous cake.*

Impossible figures—the Penrose Triangle, Escher's *Waterfall*, Sandro del Prete's *The Railway Bridge*, and more—make use of the same ambiguity and shifting of viewing location, depending on which portion of the figure we are currently looking at and making sense of.

Ambiguous solids

Ambiguous figures such as those in Figure 6 are planar images that we could interpret as three-dimensional objects in a variety of ways. *Ambiguous solids*, in contrast, are three-dimensional objects that appear from different directions to be entirely different objects. Often an artist portrays the solid both directly and also in a mirror so that viewers can see both aspects at once.

Such ambiguous solids have a rich history. Hopkins and Evans [8, p. 88] describe a statue shown at an 1880's exhibition in the Champ de Mars "that attracted much attention from the visitors". From the front, the statue appeared to be the famous operatic character Marguerite, but the mirror reflection appeared to be a statue portraying the tragic Faust. The cover of Hofstadter's 1977 cult classic *Gödel, Escher, Bach* [7] features three lights illuminating a pair of suspended objects, each with three different shadows taking the forms of the three letters G, E, and B. Guido Moretti's 1997 sculpture "Cube to Non-cube" [17, p. 81] is a sculpture that appears to be a flat Necker cube when viewed from one direction, a flat Penrose triangle from another, and an abstract 3D sculpture from most other vantage points.

While it is relatively easy to imagine how to create the objects above, other ambiguous solids can be much more perplexing. The Japanese engineer/artist Kokichi Sugihara has devised a series of ambiguous solids that attract both academic and popular attention, featured many places, including the Neural Correlate Society's *Best Illusion of the Year* contest. His *Ambiguous Cylinders* appear to be circular or square cylinders, disjoint or interlocking, depending on whether we're looking at the object or its reflection. His "Ambiguous Garage Roof" likewise appears to be either a domed roof or a zigzag roof, simultaneously. Figure 7 shows the author holding a cylinder made from a Sugihara-inspired template. (We're indebted to Richeson [16] both for the template and the analysis).

Figure 7: *The author holding a cylinder made from Richeson's template. On the left, the cylinder appears to be circular from the viewpoint of the camera and square from the viewpoint of the reflection; on the right, turning the object by* $90°$ *shows us how the height of the cylinder varies.*

How could we design such a solid? If we position the camera and its mirror image along an east-west horizontal axis, as in Figure 8, a top-view shows us that the footprint of the actual cylinder lies halfway between the two "perceived" cylinders; the varying height of the actual cylinder can be determined by the sideview analysis, assuming that the perceived cylinders are a constant, equal height.

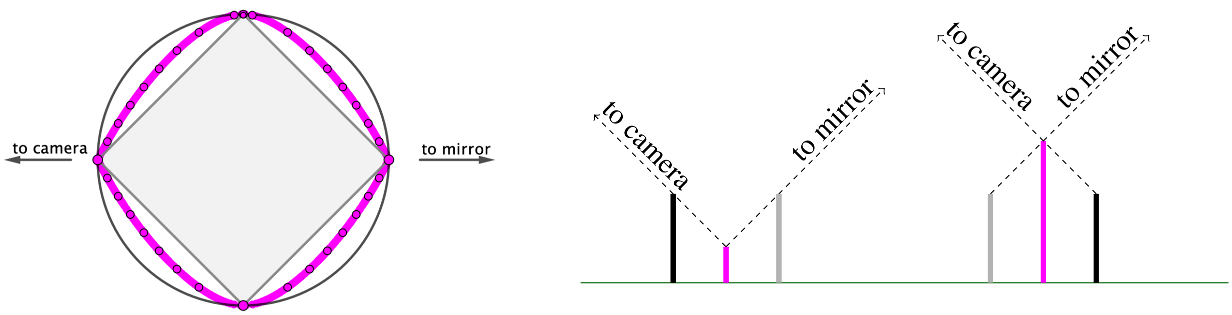


Figure 8: *Top view (left) and slice of the side view (right) showing the perceived square cylinder in gray, the perceived circular in black, and the resulting "ambiguous cylinder" in pink.*

Non-planar Canvases and Reverse Perspective

If planar canvases privilege one viewpoint over all others, non-planar canvases invite viewer movement to make their illusions come alive. Dick Termes paints his eponymous *Termespheres* [18] as though he were in the center looking outward. Viewers who look at a slowly spinning sphere from up close seem to see the world revolving around them in the opposite direction. Similarly, a number of artists make use of piecewise planar canvases that create the eerie sense that some or all of the canvas is moving as the viewer the changes position: a few notable examples include *Thinky the Dragon* [1] (unveiled at a 1993 Gathering for Gardener), Patrick Hughes' *Reversperspective*, and Roy Lichtenstein's *House I* in the National Sculpture Gallery.

An excellent example of such illusions comes from emerging artist Wendy van Boxtel [19]. Her work is a mixture of sculpture and canvas, often featuring statues climbing into or falling out of a picture frame. Several of her pieces make use of the "hollow mask" illusion, as in Figure 9. The face in her *Wilhelmina* is carved back into the plane of the picture. As we move around the face in the frame—left or right, up or down—the woman seems to turn her face toward us, or even beyond us. But it is the viewer, not the work of art, that moves.

Figure 9: *Wendy Van Boxtel's* Wilhelmina*, photographed from above, left, center, right, and below.*

How does this work? As we move (say) left, we see less of that side of the face because our angle of view approaches the angle of that side of the face. But we perceive the face coming out toward us rather than receding into the canvas, so we interpret it as a face with that side turned away from us, as in Figure 10.

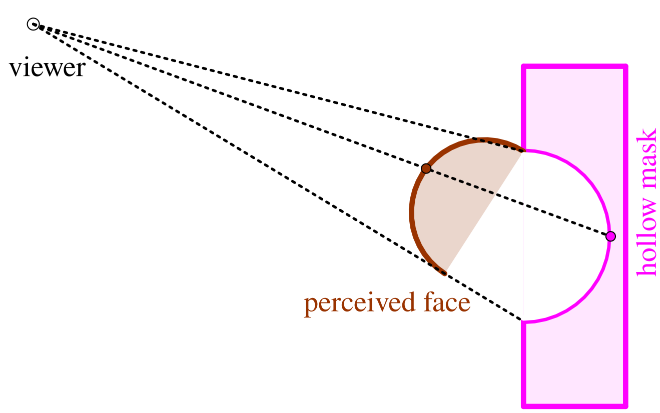


Figure 10: *If we see less of one side of a face, we perceive that face as turning toward us.*

Summary and Conclusions

A variety of illusions rely on viewers' roles in observing perspective-based art. Even straightforward perspective drawings presuppose a specific viewing target and distance; viewers who move far from the corresponding viewpoint will see the images become less realistic or even grossly distorted. Artists can make creative use of this phenomenon by making the viewpoint even more important (as with anamorphism or forced perspective). Adding mirrors allows further illusions that hide or reveal important phenomena, or even create ambiguous interpretations via multiple views. Non-planar canvases further multiply the ways in which viewers, art, and perceived images can interact. And even this roster of effects merely scratches the surface, leaving undiscussed a wealth of further illusory perspective techniques (stereoscopy, moon tilt, optically animate art, thaumatropes, pseudoscopes, and more). Exploring these illusions deepens our understanding of the geometry of perspective. Any way we see them, perspective illusions deserve another look.

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