A Woven Klein Quartic

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Abstract

We describe a new method of weaving a model of the Klein quartic, a highly symmetric, but abstract genus-3 surface akin to a platonic polyhedron, with negatively-curved geometry, based on a tiling found by G. Westendorp [10]. The Klein quartic cannot be realized in its fully symmetric form in three-dimensional space, but this model exhibits the most rigid symmetry that is possible. With remarkably little time and material you can have a Klein quartic model of your own!

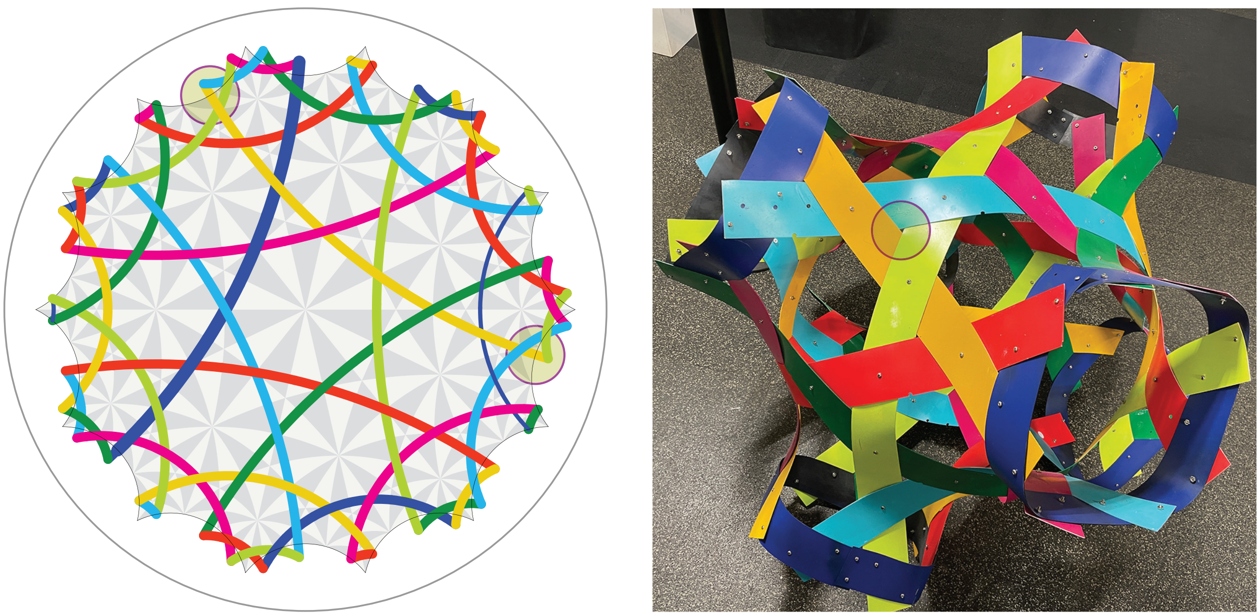

Figure 1: *The Klein quartic can be formed from a region of the hyperbolic plane with underlying *732 symmetry, shown at left, by joining like triangles on its boundary, or from the model at right, which is tiled by heptagons and triangles and bounded by geodesics; abstractly we identify opposite blue rings so that the colored strips continue. (At left, the highlighted pair of triangles are joined as one, and correspond to the highlighted point on the photograph.)*

The Klein Quartic

The Klein quartic has a rich history and can be described in many ways, touching many areas of mathematics, as beautifully represented in *The Eightfold Way*, a collection of essays edited by Silvio Levy [6]. Models and drawings of it date to its discovery. See [8, 9] for quilted and sculpted examples and much discussion.

The original representation of this surface is as a *quartic*, the solution set of a fourth-degree algebraic equation in a particular abstract space.[1] It happens that this solution set has genus-3 – it is a topologically

[1] Specifically, the solutions $[x, y, z]$ in complex projective 2-space to $x^3y + y^3z + z^3x = 0$.

equivalent to a three-holed donut. More that that, it has a rich set of 168 orientation-preserving symmetries, the most possible for a surface of this genus. This symmetry group is rather famous; one of its names is PSL(2, 7) and another is PSL(3, 2). We'll call it Q here.

To the artist, discrete geometer or a low-dimensional topologist, the Klein quartic is most easily understood as a kind of regular polyhedral symmetry, not of the sphere, but as a tiling of an abstract genus-3 surface we'll denote S.

The physical model at left in Figure 2 has much less symmetry, but topologically it has 24 heptagonal facets, meeting three-to-a-vertex. Abstractly, if we may stretch and deform S, then any flag — any triple of a coinciding vertex, edge and face — may be taken to any other by some topological homeomorphism of the surface, and so this topological tiling is *regular*. Equally, we can describe the Klein quartic as a regular tiling of a genus-3 surface by 56 triangles meeting in sevens. The two tilings are dual, with the same underlying flags, and Q preserves the tilings and the handedness of the flags. The term "Klein quartic" might refer to either of these tilings, the action of the symmetry group Q on S, or any of several other related objects. [2, 6]

In his paper [5] Klein constructs the $\{7, 3\}$ tiling of the hyperbolic plane, formed by regular heptagons with 120° vertex angles, or equally, its dual tiling by equilateral triangles meeting in sevens, both with symmetry group denoted $*732$ [3]. He shows how to wrap these tilings onto S symmetrically. The group Q is a quotient of the orientation preserving symmetry group 732. The surface S is the quotient of the hyperbolic plane by the group $732/Q$. Equivalently, we can assemble the surface by gluing together marked sides of the polygon at left in Figure 1 in such a way that colored paths match up from one side to another, and triangles on the boundary of the polygon that are colored in the same way are considered to actually be the same triangle — a pair of such triangles is indicated. On the glued-up surface, the colored stripes are geodesics. Eight heptagons zig-zag across each geodesic before it closes up into a loop, the "eight-fold way" of [6].

There are three geodesics of each color, all of the same length — at left in Figure 1, the three blue geodesics are shown with different line-widths, corresponding to the three pairs of blue loops in the sculpture at the right of the figure. In the photograph, the same triangle is marked as in the rendering at left, and from there, you can walk around and verify that the structures are the same.

Interestingly, of the thirty-five ways to choose three of seven colors, only twenty-eight appear in the Klein quartic. Seven triples do not appear, in a Fano plane of the colors: Each pair of colors is in exactly one missing triple. Each triple that does appear appears twice, in opposite orientations. (Can you spot the other blue-yellow-green triangles in Figure 1?)

We are attaching the sides of a 14-sided polygon producing a surface with one fourteen-sided face, seven edges, and (checking carefully) two vertices, for an Euler characteristic of $1 - 7 + 2 = -4$. As this surface has no boundary and is orientable, it has genus 3. We have thus decorated S with a metric of constant negative curvature, so that each heptagon is genuinely equilateral and equiangular, all preserved under the action of the symmetry group Q.

Physical models must deform this geometry but we can preserve some symmetry: The model at right in Figure 2 has tetrahedral symmetry; the group 332 is a "subgroup" of Q. (Just as 332 is a subgroup of 532, it is not a "normal" subgroup of either — $532 \approx A_5$ and Q are "simple".) S. Matsumoto's fabric model [8] has a more abstract kind of tetrahedral symmetry, in the operations that preserve it.

We can also represent our genus-3 surface S naturally in our space by cutting it open along three disjoint non-separating curves. We may arrange these six newly cut-open boundaries symmetrically in space, along the usual coordinate axes with matching boundaries opposite one another. In turn a surface formed from a lattice of these units is a topological cover of the Klein quartic and separates space into two congruent latticeworks. That surface is "nearly the same as"[2] Coxeter's infinite {6,4} and {4,6} honeycombs or the

[2]This surface is "nearly the same as" the others in the intuitive sense that you can morph space taking one to another, preserving symmetry, not moving things very far. Formally, there is a small equivariant ambient isotopy taking one to the other.

Schwarz P-surface [3]. (This geometry has been woven by Alisson Martin in [7], though without the abstract additional symmetry. Also see Figure 3.)

At a glance it would not appear that twenty-four heptagons could be regularly and symmetrically placed on this surface, but fortunately G. Westendorp shows us how on his website [10], recognizing that this gives a rendering of the Klein quartic and the basis for a woven pattern.

Once in hand the example is easier to explain: Q has a subgroup isomorphic to $432 \approx A_4$. In the coloring of the woven Klein quartic (as can be verified by tracing one color of geodesic shown at left in Figure 1) there are three geodesic loops of each color. This subgroup preserves one of those colors. Cutting along these must produce boundaries that are geodesics in the physical model. Consequently they must be flush with the boundary of the unit cell — they are loops. In In Figure 6, we show the result of cutting the same coloring of S along different colored triples of geodesic — the marked points correspond to the same point on S.

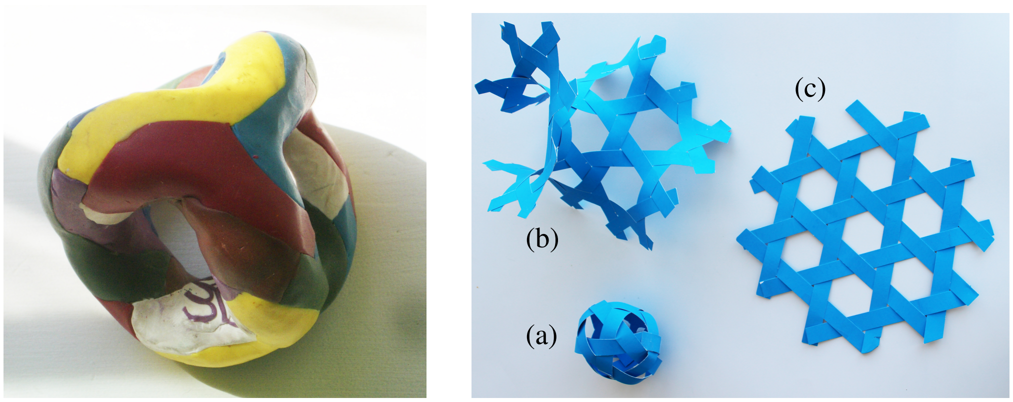

Figure 2: *At left a model of the Klein quartic with tetrahedral symmetry, formed from twenty-four heptagons meeting in threes, in eight colors. At right, controlling the surface curvature with the weaving pattern: (a) positive curvature and pentagonal holes; (b) negative curvature with heptagonal holes; (c) the kagome lattice with no curvature.*

As has been beautifully noted before [1, 4, 7] traditional (Euclidean) basketry and caning patterns can be adapted to produce surfaces of varying curvature (right, Figure 2). In the ancient Japanese kagome pattern, strips of material are woven in triangular junctures to form hexagonal holes. With pentagonal holes, we have positive curvature, the weaving of a sphere by six bands each at the equator of an icosadodecahedron. With seven, we have negative curvature, and we can realize the pattern shown at left in Figure 1 in physical space. (In the hyperbolic plane[3], and so on any surface of constant negative curvature, the angle at which strips meet is close to 58.057 degrees. There is enough "give" in the model that we may allow our strips to meet at a convenient 60°.)

On any woven surface, strips must lie along geodesics, at least to the extent that that is well-defined: any bending in the strips must be perpendicular to their normal, and so the strips remain "straight" — the strips can only follow straight paths in the underlying hyperbolic tiling.

How to Weave a Klein Quartic

Klein quartics may be woven at your choice of scale and materials: we anticipate larger, more permanent renditions of this sculpure will be woven in the future. The Klein quartic sculpture in Figure 1 is designed

[3] This angle can be measured directly in Figure 1 or by numerically solving an equation derived from the hyperbolic law of cosines.

for groups of people to come together, have fun, assembling a surprising mathematical sculpture, as a kind of communal mathematical performance art.

I made several larger plastic renderings in early 2024 that already have been assembled (and disassembled) on multiple occasions, including at *Centre international de rencontres mathemématiques* (Luminé) and the *National Museum of Mathematics* (New York). The models in Figure 3 were built by participants at the fifteenth *Gathering For Gardner* in Atlanta February 24, 2024, and show how the Klein quartic might be "unfurled" with many copies put together into a cubical grid, forming something like the famous Schwarz P surface. (However, in order for the colors to actually work out, we need a double cover of this, with two sculptures in each location of the grid!)

Those sculptures are about forty inches across; they are woven from 1/8th inch thick, 3 inch wide, spray-painted polystyrene strips, cut from sheets sold as thermoplastic, and with holes drilled from a laser-cut template. At this scale, in this material, these have a pretty bouncy feel but can hold themselves up. (A more ambitious version, twice as large but from the same material, was unable to hold its weight.) Unassembled, this version is lightweight, can be flatpacked and is easily shipped.

Figure 3: *Multiple copies of the Klein quartic assembled at the fifteenth Gathering for Gardner, 2024.*

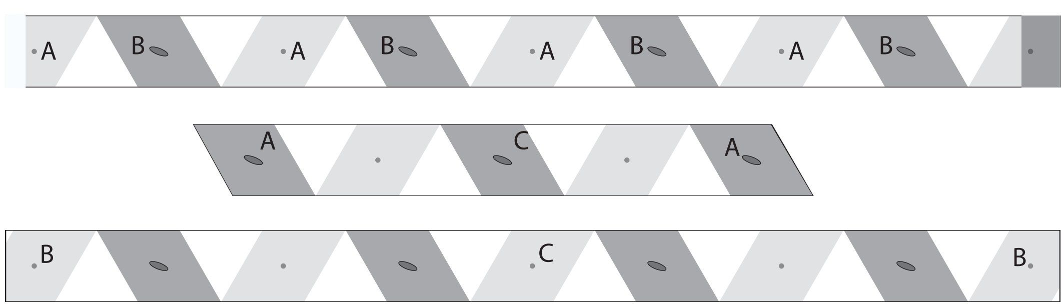

Figure 4: *Templates, at top for the "rings", middle for the "short strips", and bottom for the "long strips", available in the Supplementary file of this paper.*

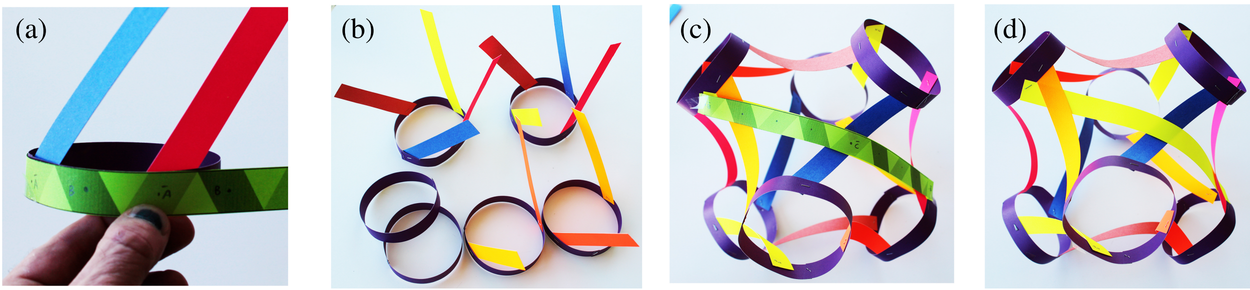

Figure 5: *Tips for constructing the model are described in the text.*

At the workshop or at home, you will find that with remarkably little time and material you can have a Klein quartic of your very own. In a workshop, colored paper strips and staplers will be provided for participants to weave their own Klein Quartic.

Instructions

1. Begin with seven colored sheets of paper or cardstock. Scaling the templates in Figure 4 by 200% fits US office paper well (the plastic sculptures were scaled by 800%), but their proportions will work at any scale. A full set of all seven ways to cut open S, as in Figure 6, efficiently uses almost all of three sheets of each of seven colors.
2. At top in Figure 4 is a template for the six "rings", all of one color, that will bound the model. Cut strips and form the rings by overlapping the left end of the template over the gray region on the right.
3. At middle of the figure is a template for "short strips" and at bottom is the template for "long strips". From each of the other six colors of paper, cut two short strips and two long strips, thirty strips all together.
4. Dark gray regions of the template will pass under another strip; lighter gray regions will pass over. The matching letters in the templates are described below.
5. As in Figure 5(a), begin by attaching four differently colored short strips to the *inside* of a ring, using the template to measure their spacing, matching A on the ring to A on a strip.
6. However that ring is colored (say ABCD), attach the other short strips of the same colors to another ring in the opposite order (DCBA) (Figure 5(b)). Then attach each of the two remaining colors to opposite sides of two of the remaining rings, as shown. (In the photo, each strip is attached to only one ring.)
7. Next, attach the loose ends of the short strips to other rings, following two simple rules: the strips and rings will form an octahedral structure, shown in Figure 5(c), with the short strips on the edges of this underlying octahedron. Second, strips of the same color will be on opposite edges.
8. We now weave in the long strips. Each end B of each long strip will weave over a B on a ring, then pass under a neighboring short strip (Figure 5(c)). A short and long strip meet at their middles C, the long strip always over the short one. The rest of the weaving will follow.

   In order to determine the color of each long strip (Figure 5(d)) note that each color of short strip meets four of the rings. The other pair of rings will be connected by a pair of long strips in that color. Because strips of the same color do not cross, this determines which way these strips must travel.

   Though the weaving will end up as over-under-over-under, etc, it may not be so during its construction! Determining which color to place next, and which way to weave each strip is a bit of a puzzle, but soon will become natural.
9. Finally, the surface S will not be complete until we glue opposite rings to one another, at least in our imaginations. Looking at opposite rings, we can see that colors continue on through the gluing, and that opposite rings are attached with a one-quarter turn twist, which we can label with arrows or letters.

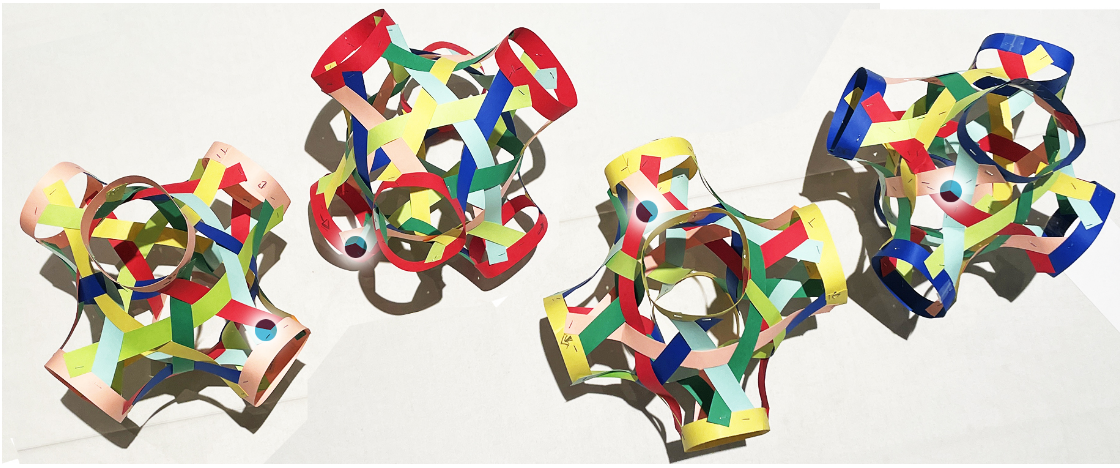

Figure 6: *These four paper models show the same arrangement of colors pattern on S, cut open along different colors of strip. The blue spots are the same location on S, the red-pink-blue triangle.*

Acknowledgements

I thank everyone who helped assemble these models over the last few months, and the reviewers of this paper for their insights and suggestions. I am especially grateful to G. Westendorp and his regular tessellation website [10] — when the insight came he had the tessellation ready — and to Eric Vergo, who helped spraypaint plastic strips for entire day.

References

[1] P. Ayres; A. G. Martin & M. Zwierzycki "Beyond the Basket Case: a principled approach to the modelling of Kagome weave patterns for the fabrication of interlaced lattice structures using straight strips." in L Hesselgren, et al.(eds), *AAG 2018. Advances in Architectural Geometry 2018*, pp. 72-93. https://research.chalmers.se/en/publication/504188.

[2] J. Baez. *Klein's Quartic Curve.* https://math.ucr.edu/home/baez/klein.html

[3] J. H. Conway, H. Burgiel and C. Goodman-Strauss. *The Symmetries of Things.* CRC Press (2008).

[4] P. Gailiunas. "Kagome from Deltahedra."*Bridges Conference Proceedings*, 2023, pp. 337-344. https://archive.bridgesmathart.org/2023/bridges2023-337.pdf.

[5] Felix Klein. "Über die Transformationen siebenter Ordnung der elliptischen Funktionen." *Math. Annalen*, 14:428–471, 1879. (Translation into English by S. Levy [6].).

[6] Silvio Levy. *The Eightfold Way: The Beauty of Kleins Quartic Curve.* Cambridge Univ. Press, 1999.

[7] Allison Grace Martin, "Superficial Study (1) Intertwined Labyrinths." *Bridges 2013 Art Exhibition Catalog*, https://gallery.bridgesmathart.org/exhibitions/2013-bridges-conference/alison-martin

[8] E. A. Matsumoto. ""Quilting the Klein Quartic." *Bridges Conference Proceedings*, 2017, pp. 411–414. https://archive.bridgesmathart.org/2017/bridges2017-411.pdf.

[9] C. H. Séquin "Patterns on the Genus-3 Klein Quartic." https://archive.bridgesmathart.org/2006/bridges2006-245.pdf

[10] G. Westendorp. *Platonic tilings of Riemann surfaces.* https://westy31.nl/Geometry/Geometry.html