

# Bend-a-Ball: Make Flexible Zonohedra and Other Polyhedra with a Simple Modular Origami Edge Unit

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## Abstract

In this workshop we will make modular origami structures that bend and collapse, commonly known as flexiballs. However, instead of Jorge Pardo's unit, we will use an improved and simplified unit that makes for better results. Within the 90 minutes of the workshop, we should be able to make the 12-unit cube/rhombohedron and/or the 24-unit rhombic dodecahedron: these are degree-3 and degree-4 polar zonohedra (PZ). 20 units make a partial icosahedron. Experienced folders may wish to begin to make the 40-unit degree-5 PZ rhombic icosahedron or 60-unit rhombic triacontahedron. Practical considerations are the number of units and desired colours. One classic colouring scheme is to use the same colour for parallel edges, another is the same colour for edges meeting at a vertex.

## Introduction

Polyhedra are natural subjects for modular origami. The Platonic solids are popular, as are some Archimedean solids [7]. Typically, the origami units act as either *face*, *vertex* or *edge* units. To make a given polyhedron, we might choose face units as this usually minimises the number of units needed. Choosing edge units usually maximises the number of units needed, so what are the benefits of this apparently more onerous approach? One benefit is the versatility of edge units: the same unit can make different angles by assembling different numbers of units at a vertex. Another benefit is the edge units can be used to make *wireframe origami*, typically polyhedral frames that interpenetrate each other, e.g. five intersecting tetrahedra and other polypolyhedra [9, 8].

However, some edge units can be awkward to assemble at a vertex: it can be like trying to close a door to a room from the outside – but only using the door handle inside the room. The flexible nature of zonohedra means that the vertex can be isolated and worked on more easily, e.g. see the right hand tip of Figure 1.

Furthermore, the vertices of edge units can flex leading to a form of action modular origami [5, 6].

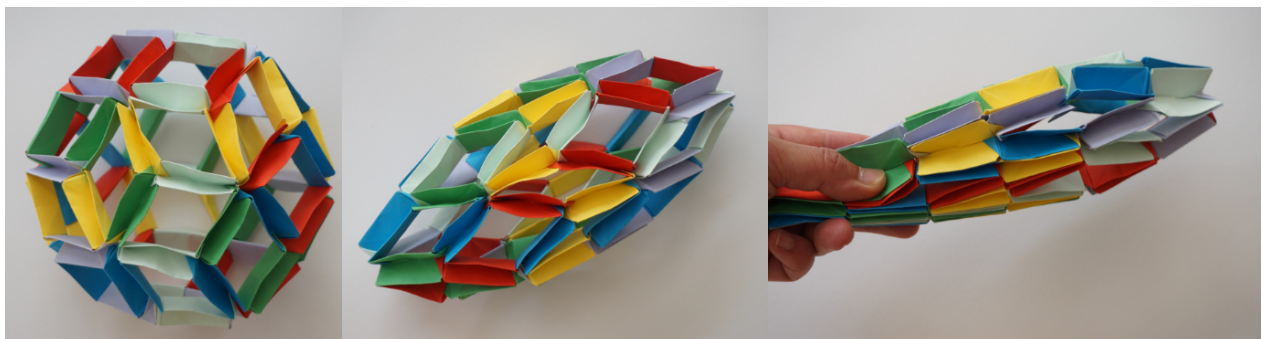
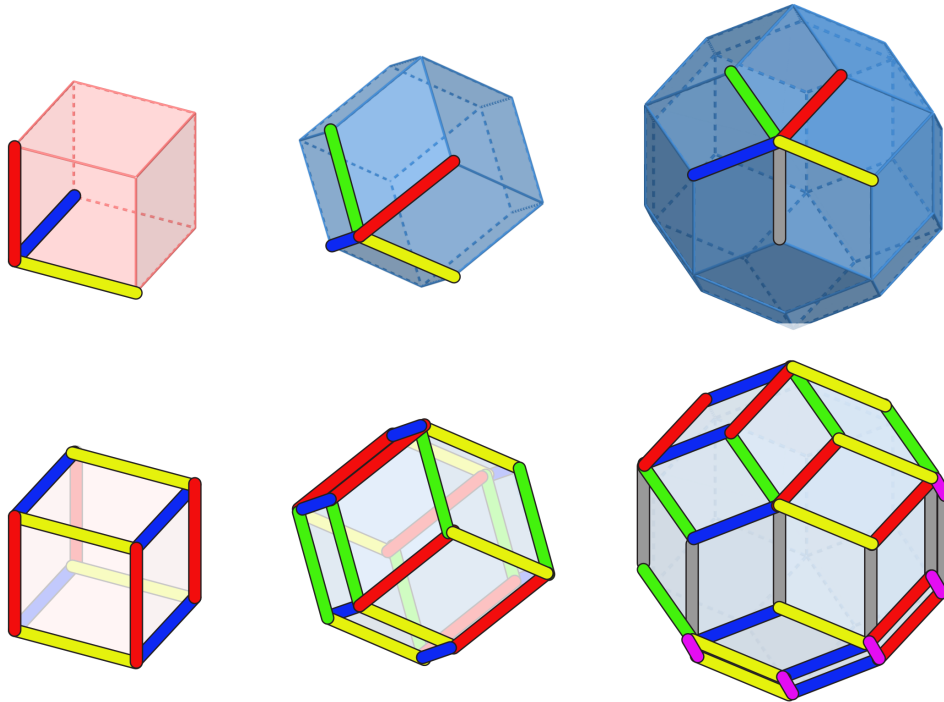


Figure 1: Rhombic Triacontahedron collapsing

## Number of Units and Colours

First, choose the number of units that you can make in the given time. Within the workshop, you should be able to make at least 12 units and have sufficient time to assemble the cube (Figure 2). You may have enough time to try the 24-unit rhombic dodecahedron or the 20-unit partial icosahedron (Figures 9 and 10).

A simple colouring scheme is to make all units the same colour. A little more sophisticated is to use the same colour for edges meeting at some vertices. However, I feel it is most instructive to make parallel edges the same colour (and sometimes the most challenging). Here are the numbers needed for three most appealing constructions:



**Figure 2:** *Using the same colour for parallel edges.*  
 Left: cube. Middle: rhombic dodecahedron. Right: rhombic triacontahedron.  
 Top row: each edge of a star is a different colour.  
 Bottom row: opposite edges of each rhombus are the same colour.

### **Cube**

Make 12 units in three colours. All vertices have three edges meeting. For convenience, use three sheets in different colours. Before cutting each sheet into quarters, divide the short edge into eighths: this precreasing makes the first few module steps easier.

### **Rhombic Dodecahedron**

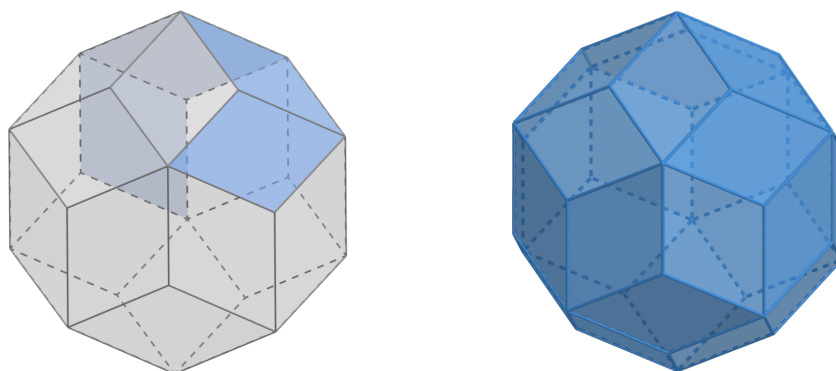
Make 24 units in four colours. Vertices have either three or four edges meeting.

### **Rhombic Triacontahedron**

Make 60 units in six colours. Vertices have either three or five edges meeting. Think of the starting vertex as the north pole: the sixth colour appears at the equator and the south pole has the colours of the north pole in reverse order.

### Zonohedra

The cube and rhombic dodecahedron are 3-fold and 4-fold polar zonohedra (PZ), respectively. However, the rhombic triacontahedron is a zonohedron but not a PZ: a 5-fold PZ is a rhombic icosahedron (Figure 3). For a rhombic icosahedron, 5 faces meet at the poles, 3 meet at the vertices next to the poles and 4 faces meet at all other vertices. A zone is the set of faces sharing one edge direction, i.e. the edges are parallel. Half of a zone is highlighted below left. Table 1 shows the number of faces, edges and vertices of selected PZ. The number of distinct face shapes is, at most,  $n/2$  rounded down to a whole number [3].



**Figure 3:** *Left: rhombic icosahedron, a polar zonohedron of degree 5. Right: rhombic triacontahedron, a zonohedron but not a polar zonohedron.*

**Table 1:** *Number of faces, edges, vertices and distinct faces for polar zonohedra of degree 3 to 20*

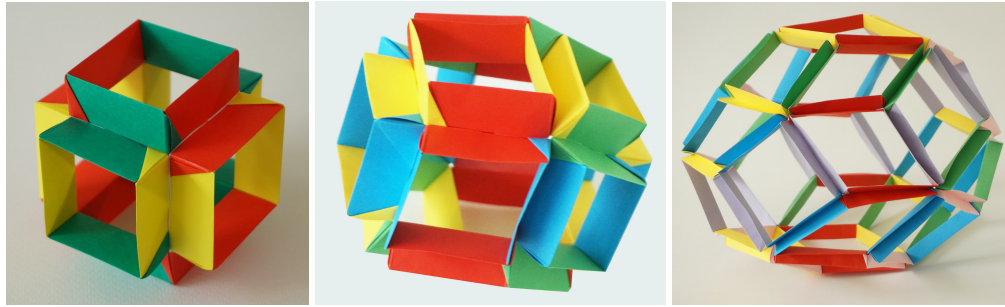
Degree $n$	Faces $F$ $n(n-1)$	Edges $E$ $2n(n-1)$	Vertices $V$ $n(n-1)+2$	Distinct Faces $F_d$ $\text{floor}(n/2)$	Edges per Degree $E_n$ $E/n$
3	6	12	8	1	4
4	12	24	14	2	6
5	20	40	22	2	8
6	30	60	32	3	10
7	42	84	44	3	12
8	56	112	58	4	14
9	72	144	74	4	16
10	90	180	92	5	18

### Materials and Optional Cutting Plan

The unit works equally well from squares or oblongs. Longer rectangles make units with longer edges. However, depending on the paper, oblongs longer about 2:1 or 3:1 can be weak and can buckle in the middle. Ideal proportions are between 1:1 and about 2:3. Figure 4 shows examples using square and long rectangles. The large assembly using long rectangle tends to collapse under its own weight. However, the small assembly using squares is stronger.

Two approaches are possible:

- Fold one sheet of paper for each unit. Use memo cube paper as a convenient source of colourful squares: the edge length is usually around 10 cm.
- Alternatively, prepare larger sheets and cut into smaller sheets for folding.



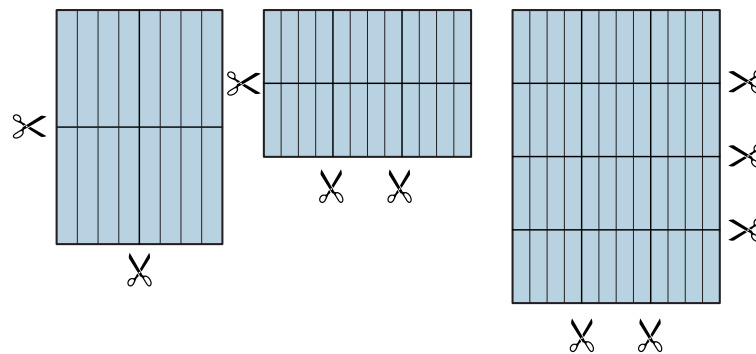
**Figure 4:** Left: 12-unit assembly using 75 mm origami paper. Middle: 24-unit assembly using medium oblongs. Right: 60-unit assembly using long rectangles of printer paper.

When starting with larger sheets like A4 or letter size paper, the simplest and most accurate method for cutting is to divide each sheet into either four, eight or 16 rectangles per sheet. However, six or twelve rectangles are almost as convenient and may be needed when you only have a limited amount of colour paper (Figure 5). Although scissors indicate which lines to cut, more accurate results come from folding along the line and slitting with a knife or certain kinds of paper cutters (typically made for opening envelopes).

You can precrease the folds used in step 1. For example, before cutting four sheets from one sheet, divide the short edge of the original sheet into eighths. For six sheets from one, divide the long edge into twelfths. For twelve, divide the short edge into twelfths. Use valley folds in all divisions.

You can also prepare the crease in step 5. However, the accuracy may be worse than folding each unit separately.

Note that machine-made paper has grain, which is usually parallel to the longer edge of the oblong. Try to align the first valley folds along the grain for stronger units. The effect is more noticeable for longer oblongs, say  $> 1.5:1$ . However, longer units can be easier to assemble as the two ends interfere less with each other. They also make some assemblies more effective, e.g. the partial icosahedron (Figure 9).

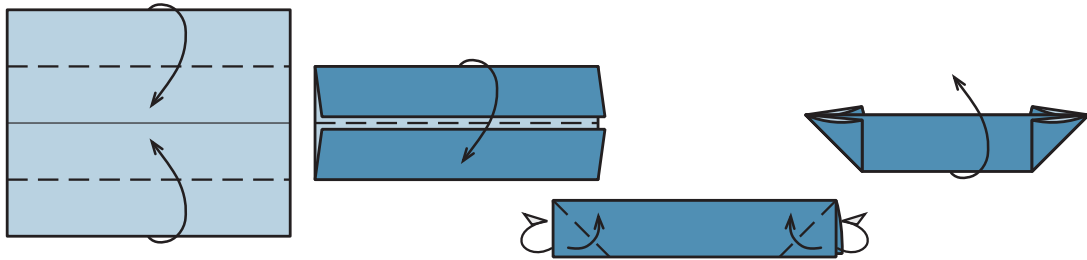


**Figure 5:** Cutting four, six and twelve rectangles from an oblong, e.g. A4 or letter size paper

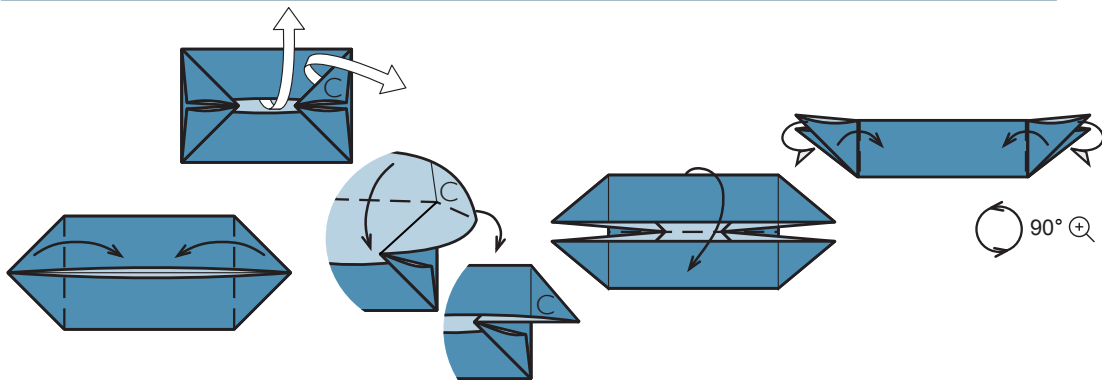
## The Module

Figure 6 shows how to fold a unit and Figure 7 shows how to assemble three units at a vertex. The unit arose whilst trying to create a different 30-piece assembly: I had found a joining mechanism to make structures like Jorge Pardo's *Flex-a-ball* [11, 10] using a simpler join, yet still as effective. The units can be reused to make other shapes since the units are easy to disassemble. The joint is related to that used by Tomoko Fuse in her *Action Lizard* [1, p. 34].

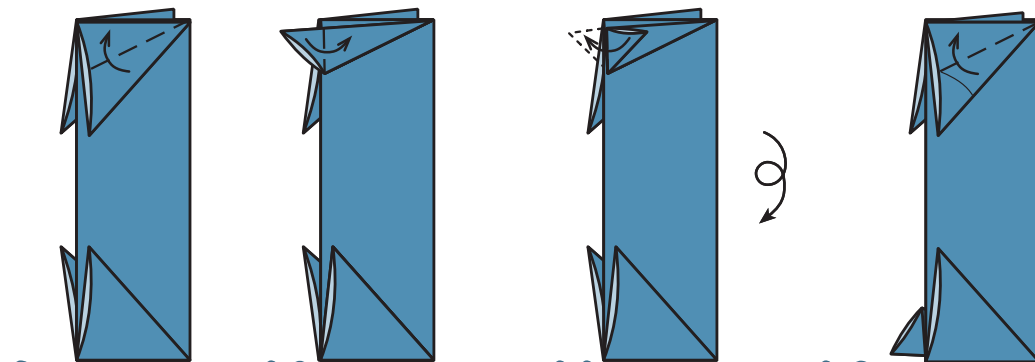
**Folding the unit**



- 1** Use a 10 cm square or oblong. For the 5:4 oblong here, fold the longer edges towards the middle. Leave a tiny gap, say 2 mm.
- 2** Fold in half, bringing the top down.
- 3** Fold all four corners up.
- 4** Open up, lifting the flap upwards so that the folded corners are underneath.



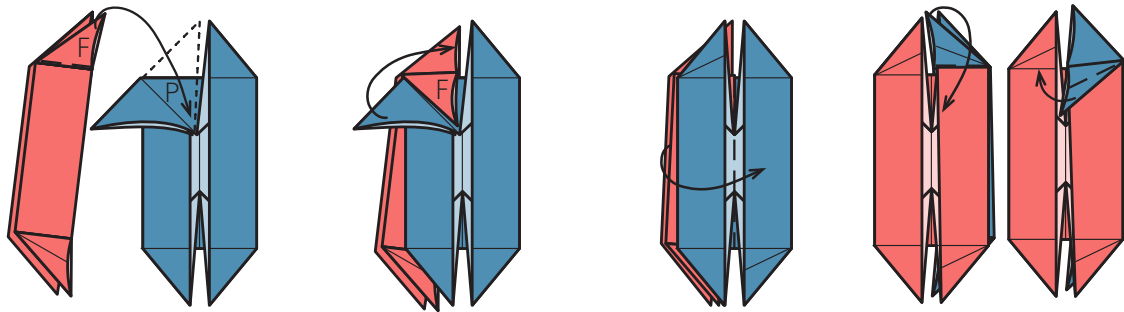
- 5** Fold the ends in along the edges of the inner rectangle: the folded corners will appear from behind.
- 6** Pull out the top right corner C, partly unfolding the model. Fold C to the right and flatten.
- 7** Repeat step 6 on the other three corners. Then fold in half bringing the top down.
- 8** Fold the four flaps towards the centre. Rotate 90° anticlockwise.



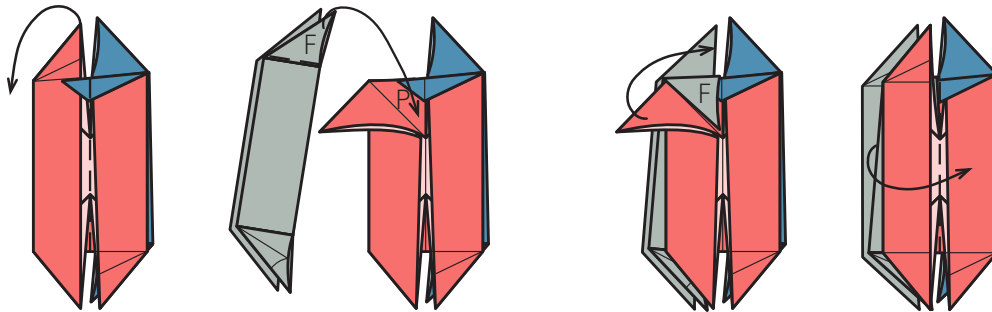
- 9** Fold the top flap: bisect the 45° angle but leave a gap about 1 mm.
- 10** Fold the tip of the flap right.
- 11** Unfold the tip. Then flip the whole model over, top to bottom.
- 12** Repeat steps 9 and 10 once more to complete the unit. This step is applied to only two of the four flaps.

**Figure 6:** *Folding the unit*

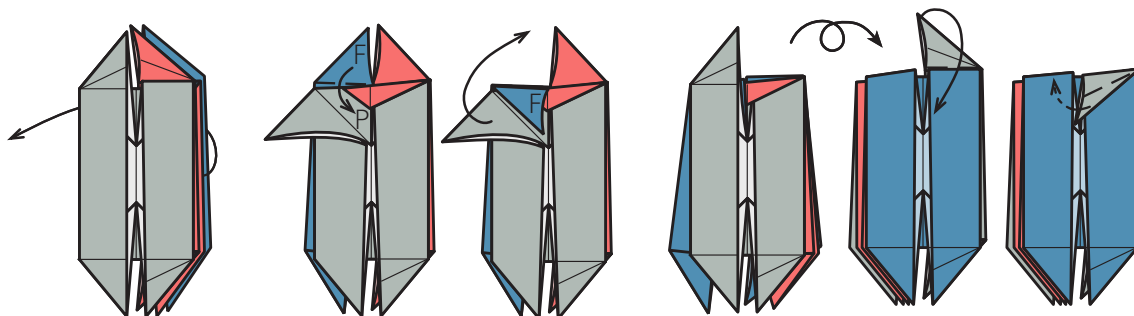
### Assembling a Vertex



- 1** Bend the top left flap of the first unit to show pocket P. Fold flap F of the second unit down into pocket P.
- 2** Return the flap of the first unit to its original position: this covers flap F of the first unit.
- 3** The uppermost left flap has joined the first and second units. Fold this to the right.
- 4** Then fold down the top flap of the first unit and fold it along its existing crease.



- 5** Bend (but do not fold) the top left flap of the second unit to reveal pocket P.
- 6** Fold flap F of the third unit down into pocket P. F covers the folded tip of the first unit.
- 7** Return the flap of the second unit to its original position: this covers flap F of the third unit.
- 8** Repeat steps 3 to 7 with any extra units needed (not shown for this example which joins three units).



- 9** To finish the vertex, fold the rear flap of the first unit to the left.
- 10** Bend the top left flap of the third unit to reveal pocket P. Fold flap F of the first unit down into the pocket.
- 11** Return the top left flap of the last unit to its original position. Turn over.
- 12** Use the existing creases to tuck the tip of the flap of the last unit into the first unit.

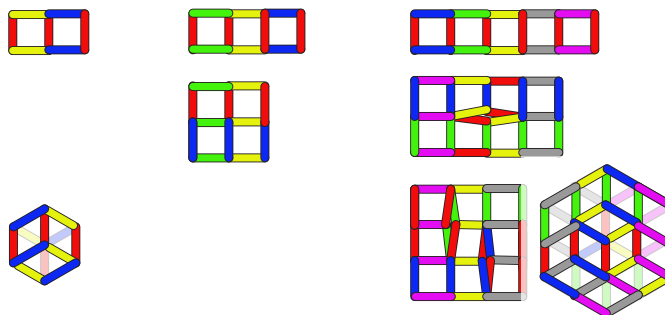
**Figure 7:** Assembling three units to make a vertex. See Figure 4 for the completed vertex in a cubic form. This method generalises to four, five and more units.

### 3D Assembly and Colours

To make these flexible assemblies, start with a star of three, four or five colours (Figure 2). Opposite edges of each rhombic face are the same colours. All edges of the same colour are parallel and form a belt that loops around the polyhedron.

#### *Collapsing*

Each assembly can collapse in many ways. Figure 8 shows some notable configurations. What other shapes can you find?

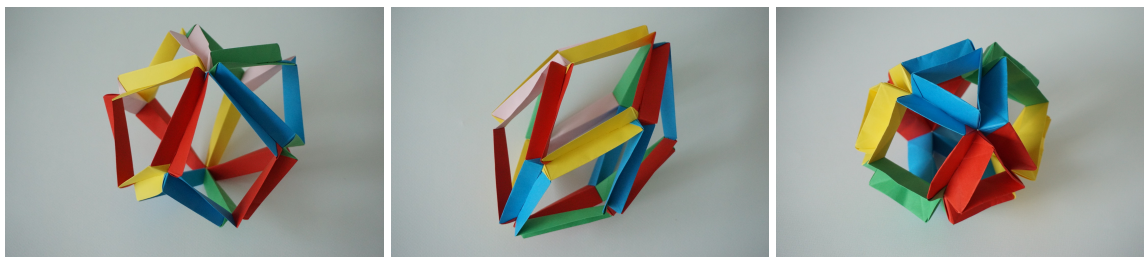


**Figure 8:** *Collapsing the cube (left), rhombic dodecahedron (middle) and rhombic triacontahedron (right) can make rectangular or hexagonal arrays of squares or triangles.*

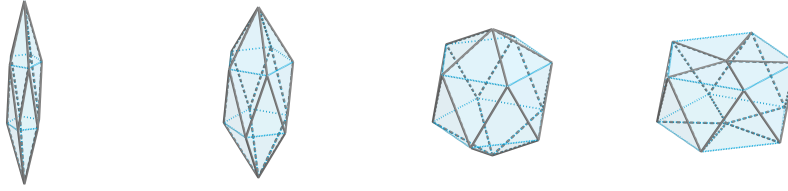
### Other 3D Assemblies

20 units make a partial icosahedron (Figure 9), a generalisation of the rhombohedron. Pulling the poles apart makes a line and pushing them together makes a pentagonal, or degree 5, antiprism (Figure 10). Use longer rectangles for a more effective assembly. In general, an assembly of degree  $n$  needs  $4n$  units. Note that 30 units make a rigid icosahedron that will not flex.

24 units make a cuboctahedron but this not a good candidate for the Jitterbug transformation [2]. The two causes are the mutual interference of the projecting flaps of paper and the constrained movement at the joints (not enough excess paper). A more successful origami approach is to make one origami unit per vertex [4] [6, pp. 58–60] [7, pp. 110–113].



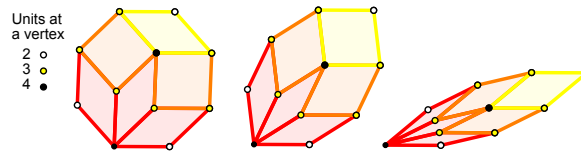
**Figure 9:** *The partial icosahedron (left and middle) made from 20 units will flex and change shape. However, the cuboctahedron (right) does not flex very well and is not a good candidate for the Jitterbug transformation.*



**Figure 10:** The partial icosahedron transforming into a pentagonal antiprism

## 2D Assembly

Curiously, 2D structures can be harder to assemble than the 3D structures (Figure 11). They also do not flex as well. Reasons include the need for units to be fully folded in half and the fact that two units do not join well together.



**Figure 11:** A 2D assembly of 16 units. The octagon is a zonogon that can collapse into a line.

## Acknowledgements

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