Brunnian Ti-Links: Using Tiles to Generate Finite and Infinite Brunnian Links

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Abstract

While the usual goal of tying knots is to hold things robustly, Brunnian links are the other extreme: removing a single component will cause the link to unravel. This interesting property makes these delicate configurations unique amongst links. While there are sporadic finite constructions, and a few infinite families, few methods are known for constructing such links. In this paper, we briefly survey previous work on this topic, then introduce a novel system of generating Brunnian links based on decorated tiles. This yields a vast array of finite and infinite Brunnian links.

Introduction

Imagine you wanted to do a really bad job of hanging a picture. Rather than passing the wire over both nails, you want to string the wire around each nail such that if a single nail came loose, then the painting would fall. This is known as the Picture Hanging Puzzle, and solutions are intimately related to the concept of Brunnian links [4].

Formally speaking, a k component "link" is an embedding, or injective function, from k circles into $\mathbb{R}^3$, up to continuous deformations [2]. Intuitively, these are the result of entangling k strings, then joining the ends of each. They represent the same mathematical structure when moved smoothly, so long as they are not cut or glued. A "Brunnian" link with k components has the property that every $k-1$ component sub-link is unlinked [9, 8]. This means removing any component allows the rest to be disentangled. Consequently, these links are quite fragile; only the intricate interplay between the components allows them to cohere.

The simplest nontrivial example of a Brunnian Link is a three component link known as the Borromean rings, Fig. 1(a). A diagram of this link has some nice rotational symmetry, perhaps this is why it is featured widely in art. This was used as the coat of arms of the Borromeo family, hence it's name. It has been used in numerous designs and has appeared in various cultures throughout history [6]. Note that indeed any pair of links are not entangled, so indeed this has the Brunnian property.

There exist multiple distinct Brunnian links with n components for all n. One classic construction is called the "rubber band" construction, as it can be made by sequentially looping circles through the opening created by the previous band, Fig. 1(b). This can then be closed off with a single loop at each end, or the first and last loops can be linked, forming a rotationally symmetric design. This is the basis of some children's toy looms that use elastic bands, such as the Rainbow or Wonder Loom products. Similarly, a few chainmail jewelry designs are also Brunnian, for example the "fox tail" or "wheat" patterns, though the vast majority interlock the metal rings for better structural integrity. It happens that the Borrommean rings are the base case of this construction.

A related infinite family is made by a series of square knot links, as in Fig 1(c), [5, 11]. This construction is somewhat nicer, as each link only contributes six crossings to the total, so it is a bit simpler. In contrast, each link in the rubber band construction adds eight crossings. There is also an interesting example of a two dimensional infinite Brunnian link [1].

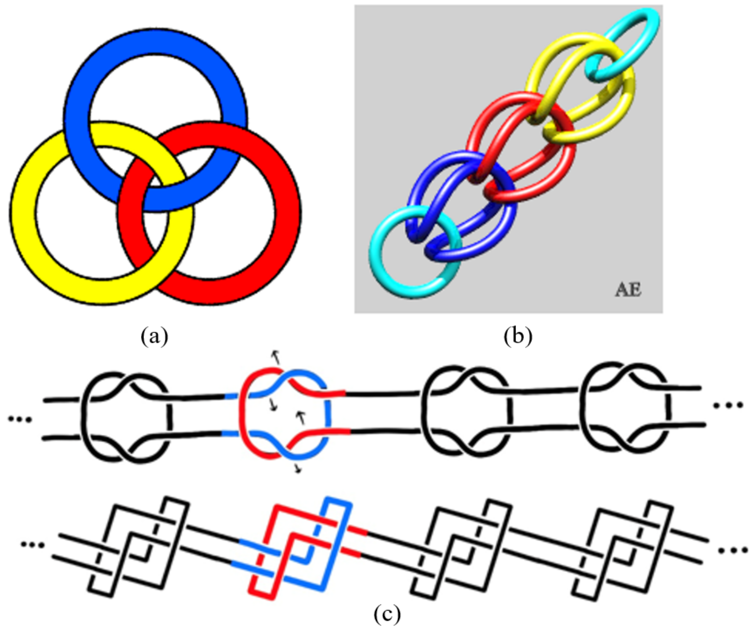

Figure 1: *(a) The Borrommean rings [6]. (b) An example of the rubberband construction, which can be extended arbitrarily [7]. (c) a section of the square knot, or lark's head, construction [5]. Both constructions can be made rotationally symmetric by connecting the first and last bands.*

This paper generalizes another presentation of the square knot construction to create a broad array of designs, Fig 4. To the best of the author's knowledge, this system has not been considered previously. However it is difficult to be certain since the literature is quite dispersed.

The Brunnian Ti-Links

Inspired by viewing the square knot construction as a sequence of pairs of band ends "grabbing" each other, Fig. 1(c), we extend this to multiple bands looping through each other in a circular fashion, Fig. 2. This is reminiscent of some logos of a circle of arms where each grabs the wrist of the next, and windmill loops in weaving paper strips [3]. By placing these in a regular polygon of the same number of sides, with the bands connected to the midpoints of each edge, tiles can be joined to form a design, Fig. 3 (a). Because the bands pass through the midpoints of each edge, any tiling by regular polygons can be made into a link by using the tiles from Fig. 2, once the free pairs on exterior edges of the tiling are connected. Additionally, the insides of each tile may be mirrored, independently of each other, adding further design possibilities. Since the resulting links are based on tiles, we have named them with the portmanteau "Ti-Links."

If there is an even number of free edges for some finite tiling with boundary, essentially a flattened tiling of a sphere, they can be connected without introducing a knot with the resulting link being Brunnian. For example, one may connect adjacent pairs of free ends, Fig. 3 (a), or loop each through the next in the same fashion as the center of any tile. If any band is removed, all other bands it wrapped around are free to retract through the edges of their tiles, which effectively removes them from the other bands in the adjacent time, Fig. 3 (b). Each retracted band can then disentangle from the other bands in it's tile, leading them to retract through all other edges of the tile. Since every tile is connected, this retraction propagates throughout the tiling, ultimately unlinking every component, Fig. 3(b). This shows that any sublink is trivial. To show the whole link is itself nontrivial is a simple application of the Arc-method of Bai and Wang [2]. Taking the interior of a single band as the sole test link, we see there are no "ears" compressible for the link (its interior

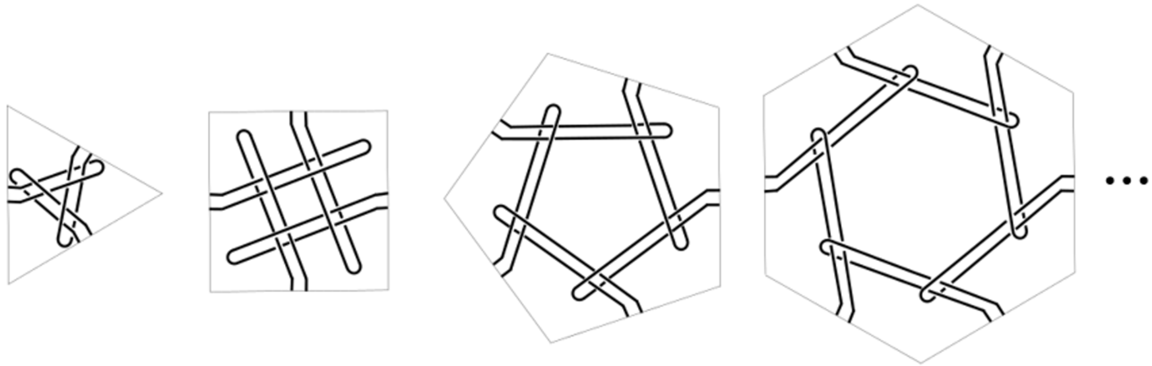

Figure 2: *The first several tiles of this paper's construction, which can be seen as an extension of the square knot construction.*

being a subset of the complement of the link), as a band passes through each. So in particular there is at most one ear that is compressible in the link, therefore by the Arc-criterion this link must be Brunnian.

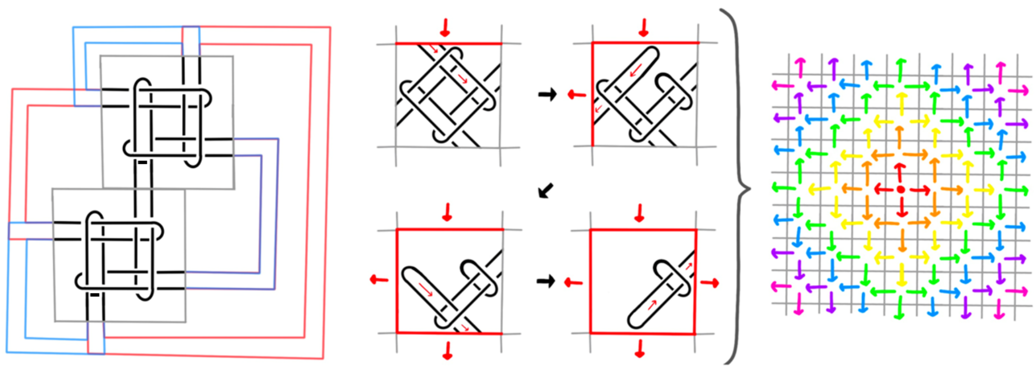

Figure 3: *(a) Two valid completions of a finite tiling, with red or blue connections added, with the overlap in purple. (b) an illustration of the Brunnian property by considering the propagation of disentangling throughout a tiling.*

This construction can be generalized further if one thinks of each circle of k interconnected bands at the center of a regular k-gon as a vertex of degree k in some graph. By substituting one of these for each vertex in a graph, where the number of bands used is the degree of that vertex, and connecting a pair of free ends for each adjacent vertex, we arrive at a Brunnian Link as well. Consequently three dimensional Brunnian meshes could also be constructed through this system, for example by using a cubic lattice. This paper takes the tiling viewpoint as it's focus because they are easier to play around with, Fig. 4, for example by making these tiles the faces of a polyhedron, as in Adam Rowe's "knot tiling triangle-faced polyhedra" [10]. The resulting polyhedral Brunnian links can be illustrated in numerous other media, for example the traditional Japanese art of Temari, Fig. 4.

Summary and Conclusions

In this paper, we give a brief survey of Brunnian links, focusing on methods of construction. The main contribution of this work is a novel approach to creating Brunnian links based on tiles, though it is substantially more general. There are a variety of patterns that can be created with variations of these tiles, which are

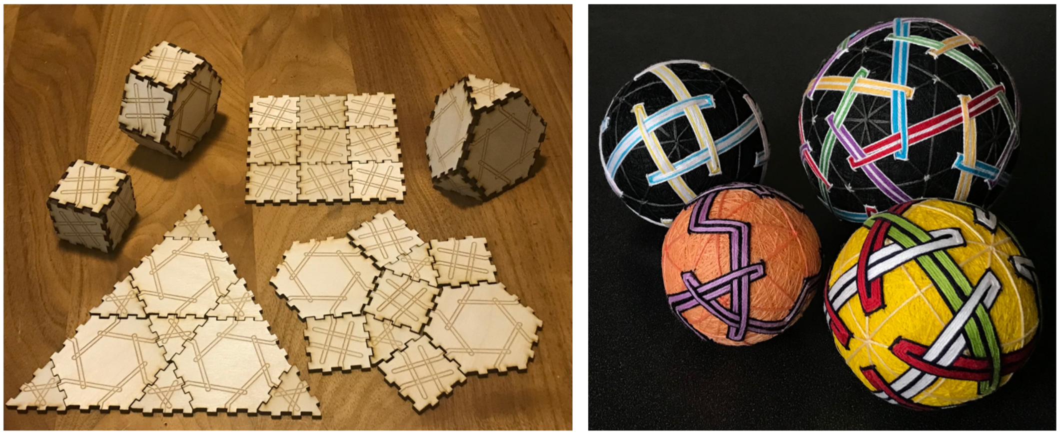

Figure 4: *Illustrations of Ti-Links made by the author. Left: Tiles laser cut from plywood, assembled into patches of tilings and polyhedra. Right: Temari illustrations of polyhedral constructions. Specifically, each corresponds to one of the platonic solids, except for the icosahedron. Each band corresponds to an edge of the associated polyhedron, so these have 6, 12, or 30 bands.*

available at the author's website. Subsequent work could address generalization of the Brunnian property, such as removing any two, but no single, band results in the unlink.

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