Unicursal Corrugated Baskets

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Abstract

Locked crossings are an innovation I introduced in basket making that allows making baskets from a spool of edge-notched ribbon, where the spool does not need to be passed through the work, and the shape of the basket is programmed by the positions of the notches. Locked crossings open a new possibility: programming the angular excess/deficit in each weave opening, thereby texturing an otherwise smooth basket into a miniature landscape of hills, dales, and saddles. I refer to this as corrugating the basket in the sense that improved stiffness is gained through patterned texturing. By reference to some results in knot theory, I explore the mathematical constraints on corrugated baskets that are made from a single, unicursal ribbon that closes into a loop. Models of the smallest such baskets, and a vase-scale maquette are illustrated.

Introduction

I introduced *locked-crossing* basket making [3] , where a ribbon-like weaving element is made to cross over itself and interlock by means of notches on both edges. When a ribbon crosses over itself (Figure 1a,) the doubly-covered region is a rhombus. Diagonals of a rhombus (dashed lines) meet perpendicularly. If, as shown in the diagram, each portion of the ribbon has four notches, with each notch tangent to a diagonal, they can engage in a way that locks the crossing. The combined effect of the notches is to make the doubly-covered region a bit smaller, but, at each corner, the region now has salients called *tangs*. The exact shape of the tangs depends on the shape of the notches and on ψ, the angle of the crossing. The rule for interlocking the crossings—a task that requires some bending and twisting—is, "over in the rhombus, under in the tangs." Notice that, if the two ribbon portions are laid side-by-side, the notch patterns are mirror images of one another.

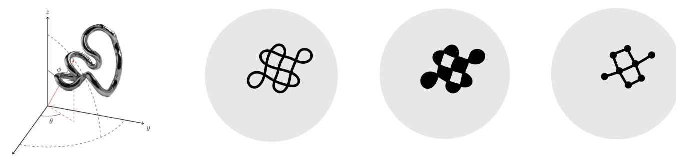
(a) **(b)** **(c)**

Figure 1: *(a) a locked crossing; (b) a sparsely woven basket with locked crossings; (c) corrugated weaving.*

One advantage of locked crossings in basket making is the possibility of weaving sparsely (Figure 1b.) A second advantage is that we can use control over the angle of crossing to corrugate the weave—raising a pattern of small hills and dales (Figure 1c) that stiffen the basket fabric—this will be our topic. A third advantage of locked crossings is that, freed from going 'over-and-under', we can go like we are winding rope on a barrel. That third advantage motivates our interest in basket structures which are unicursal, that is, made from a single length of ribbon that ultimately closes smoothly onto itself to form a loop.

Baskets from Knots

Elements of knot theory indicate how a basket can be corrugated. A *spatial knot,* Figure 2a, is a circle mapped into 3D space. Consider a spatial knot that lies in a system of spherical coordinates, (r, θ, ϕ), and does not intersect the origin, then suppressing the r-coordinate gives a *projection* of the knot on the (θ, ϕ) sphere (Figure 2b.) At this point we keep the projection and leave the knot behind. Any knot projection can

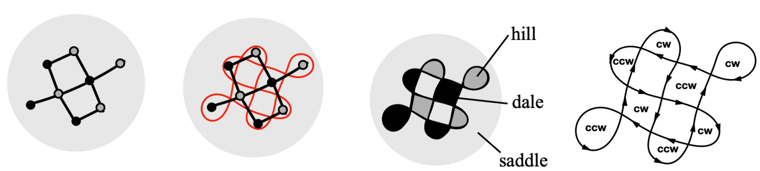


(a) **(b)** **(c)** **(d)**

Figure 2: *Knot to Tait graph: (a) a spatial knot in (r, θ, ϕ) coordinates; (b) its projection on the (θ, ϕ) sphere; (c) a chessboard coloring of the projection; (d) the underlying Tait graph.*

be given a *chessboard coloring* (a.k.a., checkerboard coloring) as in Figure 2c: each region of the projection is colored black or white such that no two regions of the same color are side-by-side neighbors. We take the liberty of calling a knot projection equipped with a chessboard coloring of its regions a *chessboard.* A chessboard can be more concisely encoded by its *black Tait graph*[1] (Figure 2d): a graph drawn on the sphere with a vertex in the interior of each black region of the chessboard and an edge for each corner-to-corner adjacency between two black regions. Following the terminology of [1], we say that a plane graph (e.g., a graph drawn on the sphere) is *unicursal* if it is a Tait graph of some knot.

We will now work in the other direction—going from a unicursal Tait graph toward a knot—but stopping at a somewhat differently colored chessboard that we will take to represent a corrugated basket. We need to start with a unicursal plane graph. How do we find one? Shank's Theorem [2] gives a characterization of unicursal planar graphs useful for computation: *a plane graph corresponds to a knot diagram if and only if the number of spanning trees of the graph is odd.* For graphs up to about 20 edges, current computers make it practical to generate all the planar graphs of a certain class and size, and then sift through them looking for spanning tree counts of odd parity. That is what I did to find the smaller examples that will be illustrated below. We will work through an example with a unicursal graph that is, for brevity, both *bipartite* and *simple,*



(a) **(b)** **(c)** **(d)**

Figure 3: *From a unicursal bipartite Tait graph to a corrugated basket: (a) a bicolored unicursal Tait graph; (b) construction of its medial; (c) its 3-colored chessboard seen as a color-coded topography; (d) an orientation of the knot projection induces the same coloring.*

and later justify those choices. A graph is bipartite if we can bicolor its vertices (here we will use two tints of black) such that no edge connects two vertices of the same tint; equivalently, a graph is bipartite if it has no odd cycles. A graph is *simple* if it contains no loops or parallel edges. A graph that is bipartite will not

[1] Tait graphs are named for Peter Guthrie Tait, a mathematical physicist who discovered this connection between graphs and knot projections in the 19th century.

contain a loop, but it might contain parallel edges. We know that the graph in Figure 3a is unicursal because we derived it in Figure 2 from a knot projection. It is also bipartite, as evidenced by its dark-tint/light-tint vertex coloring. The graph being bipartite, it suffices to establish that it is also simple by verifying that it has no parallel edges.

We now construct (Figure 3b) the graph's *medial*[2]: start by drawing a line along the perimeter of a face, but each time you reach the midpoint of an edge, "jump the fence" and continue walking in the same direction on the other side; iterate this sequence of events. Since the graph is unicursal, this "fence-jumping" tour will close on itself after crossing each edge twice. The original graph will then be deleted, but not before we have assigned its vertex colors to the regions of the medial that they occupy; unoccupied regions will remain white (Figure 3c.) We call the medial of a bicolored graph, G, after being face-colored in this way, the *color-inheriting medial* of G.

We now observe that the color-inheriting medial of a bicolored simple plane graph satisfies the constraints on a smooth topography partitioned into *saddles* and *non-saddles,* those non-saddles being further partitioned into *hills* (a hill being a face containing a local maximum of altitude) and *dales* (a dale being a face containing a local minimum of altitude.) For example, two hills cannot be side-by-side neighbors, nor even corner-to-corner neighbors, because a discontinuity of slope would arise at their intersection (likewise for two dales.) Saddles must intermediate everywhere; yet not even a saddle can be side-by-side with another saddle. One can observe that the color-code assignments specified in Figure 3c, white = saddle, light-tint = hill, dark tint = dale, satisfy the above constraints. We needed to start with a bipartite Tait graph to end up with a topography without discontinuous changes in slope, and we needed to start with a Tait graph that was simple so that no saddle region would be a digon.[3]

Programming Basket Curvature by Angular Excess or Deficit

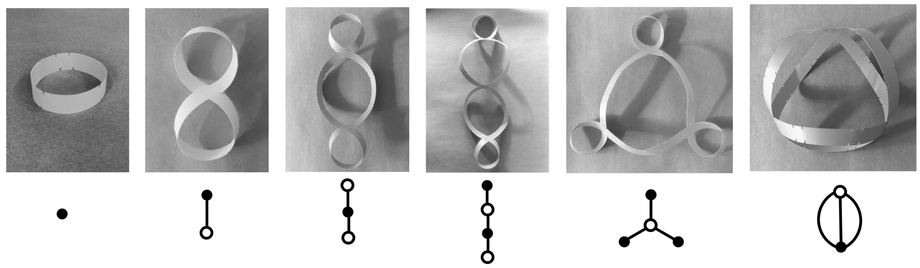
We now have a plan for distributing the two kinds of curvature on the sphere that is consistent with a bumpy (yet smoothly continuous) topography with one hill, or one dale, or one saddle, per basket opening. We can force each basket opening to assume the right kind of curvature by locking the crossings at angles that deviate slightly from the crossing angles found in the un-corrugated basket. For example, the angle sum of any quadrilateral on the plane is 360°. If such a quadrangle is realized with ribbons and locked crossings, and its four internal angles are forced be more obtuse (creating angular excess,) the structure will assume a spherical curvature; if we force the internal angles to be more acute (creating angular deficit) the structure will assume a hyperbolic (saddle-like) curvature. Of course, giving two opposite sectors at a crossing more angular extent, just robs angular extent from the other two sectors—but this is exactly the sort of trade-off the 3-colored chessboard permits: making the two tinted sectors more obtuse (thus making their respective openings more spherical) just makes the two white sectors more acute (thus making their respective openings more hyperbolic).

The Models

The models in Figures 4 and 5 use ribbons cut from 65-lb cardstock in a Cricut Maker 3, which limits length-of-cut to about 60 cm. Sufficient ribbon lengths were assembled from 60 cm segments by transparent-taping on both faces of 45° splices. The partition of non-saddle openings into hills and dales is something the weaver must keep in mind: when completing a non-saddle weave opening, hills must be forced convex, and dales must be forced concave.

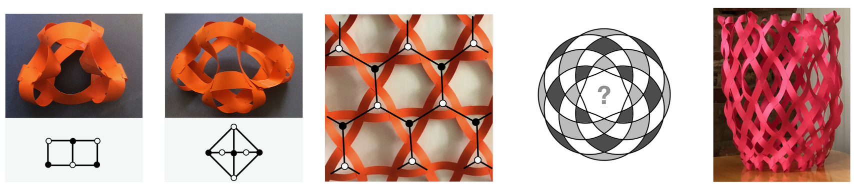
[2]For a formal definition of *medial*, see [1] Sec. 3.1.

[3]A pair of ribbons with two crossings locked to form a digon intrinsically have spherical—not saddle-like—curvature. For an example, see Figure 4f below.



(a) **(b)** **(c)** **(d)** **(e)** **(f)**

Figure 4: *The six unicursal bipartite planar graphs of 3 edges or less (a)-(f), and their corresponding (corrugated, if possible) baskets. Basket (f) has digonal openings—derived from the digonal faces in its Tait graph—preventing it from being corrugated.*



(a) **(b)** **(c)** **(d)** **(e)**

Figure 5: *Examples: (a-b) the two smallest unicursal simple bipartite 2-connected graphs shown with their realizations as corrugated baskets; (c) a passage of corrugated kagome shown in relation to its bipartite Tait graph; (d) a projection of the T(7, 8) flat torus knot shown with its un-complete-able topographic coloring; (e) maquette for a large aluminum basket based on a flat torus knot.*

Summary

I have introduced the topic of unicursal corrugated baskets and shown that they correspond to knot projections having simple, bipartite Tait graphs.

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