An Aperiodic Pied-de-Poule (Houndstooth) Tiling for e-Fashion

Loe Feijs

Laurentius.Lab and University of Technology Eindhoven, The Netherlands; l.m.g.feijs@tue.nl

Abstract

We present an aperiodic tiling which is visually like the well-known Pied-de-poule textile pattern yet is aperiodic. Although the specific tiling does not repeat, the tiles do not force aperiodicity. The aperiodicity comes from a special sequence of zeros and ones, following a 1981 paper by De Bruijn. We implement the pattern in an electronic brooch using an Arduino and an LCD display.

Introduction

Pied-de-poule, also known as houndstooth, is a weaving pattern, obtained by alternating black and white bands in both the warp and the weft. With a plain binding this produces a checkered pattern, but with a twill binding (two over, two under, e.g.), Pied-de-poule emerges. This pattern is periodic, and during the 2023 excitement around the new Hat tiling, the question occurred to me whether I could make an aperiodic Pied-de-poule. Figure 1 is an early sample (mixed embroidery/weaving).

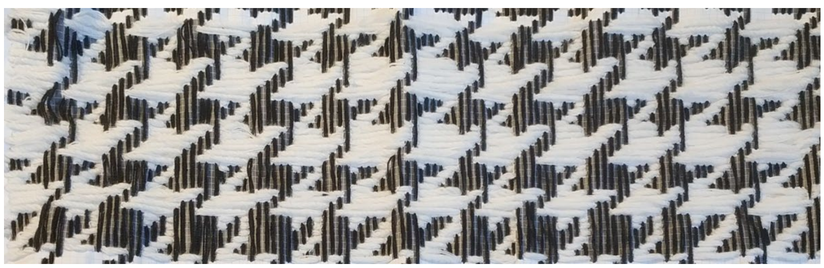
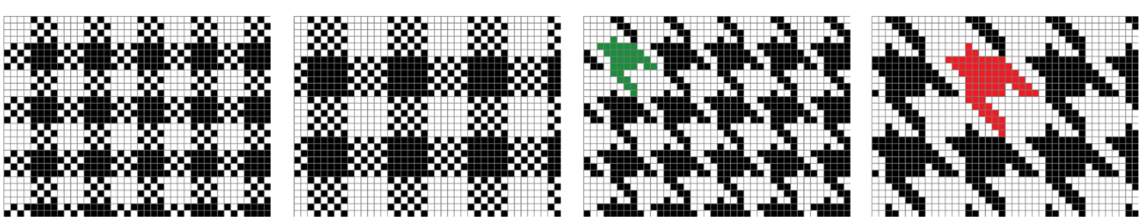


Figure 1: *Sample of an aperiodic pied-de-poule tiling, combined embroidery and weaving.*

De Bruijn has shown that there is a one-dimensional aperiodicity hidden in Penrose rhombus tilings, which unfolds in five different directions [1][2]. From this, I got the idea that an aperiodic sequence of zeros and ones by De Bruijn [2] can be used to define a Pied-de-poule [3] with variable width color bands in warp and weft. The innovation is that bands of 4 and 6 are mixed according to the zeros and ones of the aperiodic sequence. We find eleven tiles, but different from the Hat tiling [4], the tiles do not force aperiodicity.

There is not one Pied-de-poule pattern, but a family, one for each integer $N \geq 1$ [3]. The case $N = 1$ is called puppytooth. Here we focus on $N = 2$ and $N = 3$, shown in Figure 2 (c) and (d), respectively. We see black tiles and white tiles with a core block of 4×4 each, as in Figure 2 c), or 6×6, as in 2 (d). One *tile* is highlighted green in (c), red in (d). Between the core blocks we see more complicated blocks, featuring protruding tails from the core. It is the binding (or weave), which distinguishes the Gingham (or Vichy) of Figure 2 (a, b) from the Pied-de-poule of (c, d). Gingham has plain binding (one over, one under), Pied-de-poule twill binding (N over, N under). To weave an N-type Pied-de-poule, one needs a loom with $2N$ shafts.

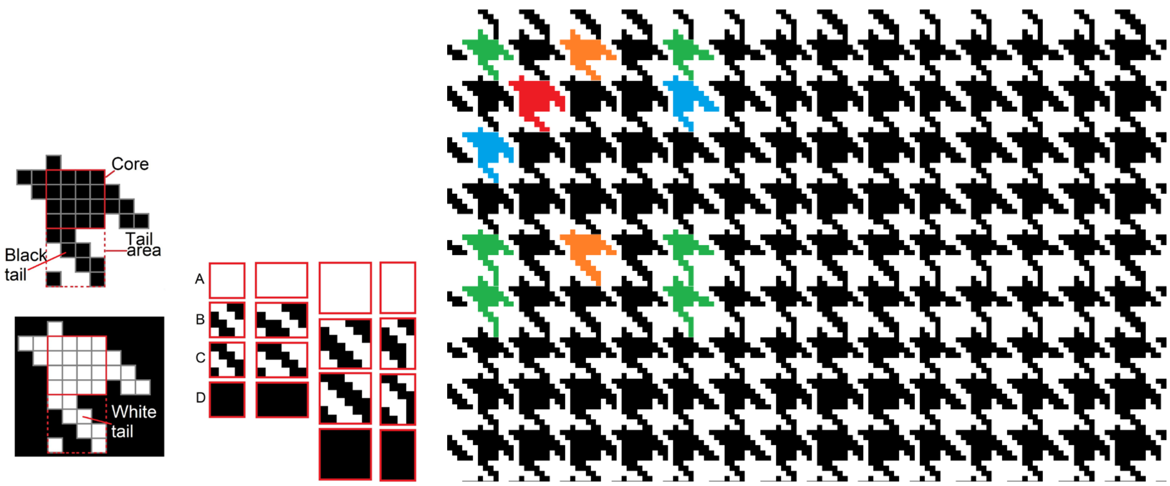


(a) (b) (c) (d)

Figure 2: *Four classical weaving patterns (a and b: checkered, c and d: Pied-de-poule).*

Here, aperiodicity is produced by including both 4×4 and 6×6 blocks in the same tiling, and where tiling of the plane is therefore facilitated by the introduction of mixed-sized blocks of 4×6 and 6×4. We need two kinds of blocks, viz. “core” blocks, which are entirely black or entirely white, and “tail area” blocks, which are mixed black/white, see Figure 3(a). We label the block types: A *white core*, B *white tail*, C *black tail*, D *black core* (Figure 3 a and b). In a white column, we alternate between A and B blocks, in a black column C and D. The block size is governed by an aperiodic sequence of zeros and ones. The same sequence is used for the vertical and the horizontal bands (one could use different sequences such as the golden ratio-based sequences of Section 10 in [2]). To design the 4×6 and 6×4 tail blocks, I used the vague criterion that the result had to be “Pied-de-poule-like”. The bands in Figure 3 (c) are 6,4,4,6,4,6,4,6,4,4,6,4,6,4,4, 6,4,6,4,4,6, etc. wide, corresponding to the sequence 100101010010100101001, etc. from [2].

As shown in Figure 3 (c), eleven distinct tiles emerge from the interplay of the 16 block types and the sequence of zeros and ones. I call them *mutants* as they look like a classical tile [3] yet are different[1]. The red mutant has a 6×6 core, the 6 green mutants a 4×4 core (2 orange 6×4, 2 blue 4×6). Why only one red? Subsequence 11 does not occur, so the red core has only one option of neighbor block on each side.



(a) (b) (c)

Figure 3: *Anatomy of variable-width tiles. (a) Black and white classical tiles. (b) Building blocks of distinct widths and heights. (c) Aperiodic tiling fragment (one occurrence per mutant type colored).*

Towards Fashion

In a Penrose tiling or a Hat tiling, the observer might recognize the famous tile(s) enforcing the aperiodicity. For my one-dimensional tiling, there is no such object to be recognized. To make the nature of the tiling visible, I should show the entire sequence. This is impossible in a finite piece of fabric like Figure 1, so I turn to a dynamic approach, where a finite window slides over the sequence. The observer can see everything — if he/she/they wait long enough. Therefore, I implemented my tiling in an electronic brooch, which can be worn as a fashion statement.

The display is a 0.96-inch OLED of 128×64 pixels. It has a two-wire I2C bus and two 3.3V supply wires connected to an Arduino microcontroller, in my case a Seeeduino XIAO, which is only 20×17.5×3.5mm. The intended fashion statement is that tech is cool, that mathematics is beautiful and that weaving patterns are cultural heritage to be preserved and rejuvenated. The brooch goes well with a men’s suit, noting that traditionally a fashionable man shows his style with subtle things (I considered cufflinks as well). No attempts are made to hide the electronic nature of the object.

[1] For an exploration into how to weave the new pattern, see the supplementary material. The exploration suggests that it takes an 8 or 10-shafts loom to weave the pattern (or a Jacquard loom). The diagonal structure of the twill binding becomes irregular. The floats have lengths 2 and 3 (unlike the manual sample of Figure 1, which has floats up to 8).

Aperiodicity from Irrationality

The sequence of zeros and ones that will determine the consecutive band widths for our Pied-de-poule is adopted from the paper by De Bruijn [2] who describes, amongst others, the so-called Beatty sequence 10010101001 .... The same sequence is described in several alternative ways. The first way is to start from 1 and repeatedly apply the substitutions $1 \rightarrow 100$ and $0 \rightarrow 10$. This way is attractive for showing similarities between Beatty sequences and Penrose tilings, notably for the theory of inflation and deflation – a kind of fractal self-similarity, sketched in [5] (the full theory is in [1][2], yet beyond the scope of this paper). This first way is easy to implement in a computer when n is not large. For finite pieces of fabric, this works fine. The second way deploys an algebraic formula that gives the n–th entry in the sequence. The formula uses the floor function, denoted by $\lfloor$ and $\rfloor$ (the largest integer less than or equal to).

$$p_n = \lfloor (n+1)(\sqrt{2}-1) + \tfrac{1}{2}\sqrt{2} \rfloor - \lfloor n(\sqrt{2}-1) + \tfrac{1}{2}\sqrt{2} \rfloor \qquad (1)$$

Formula (1) can be understood as a model of a cyclic skate track with two skaters. The track is 1km long and has a score board that shows how many full cycles each skater has completed. The starting line is at 0.707km and skaters have the same speed of 0.414km/min, but the first skater a has head start of one minute. There is a radio reporter who mentions every minute how many rounds the first skater is ahead of the second, as apparent from the scoreboard. This report thus mentions either 1 or 0.

The aperiodicity of the sequence follows by considering a plot of the line through $(0, \tfrac{1}{2}\sqrt{2})$ with a slope of $(\sqrt{2}-1)$. The ratio of the number of this line's crossings by gridlines $y \in \mathbb{Z}$ (red in Figure 4) and by gridlines $x \in \mathbb{Z}$ (blue) is a rational number for any finite segment of the line. Referring to the skater model, each red dot marks a cycle completion, each blue marker a radio report. If we try to reconstruct the entire line as a repetition of such a segment, we fail, because the slope is irrational.

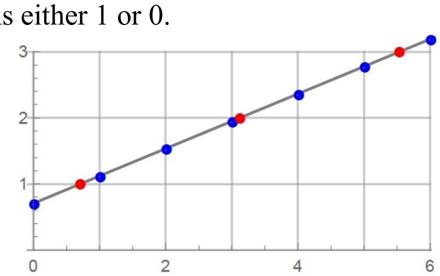


Figure 4: *Plot of a line with irrational slope and two types of grid line crossings.*

Coding

The formula (1) is useful for embedded applications, where a memory efficient manner is required to generate p_n for large values of n (the substitution method is unattractive). For given n, formula (1) can be evaluated by arithmetic. By 32-bit IEEE754 floating-point arithmetic, I get correct $p(n)$ values for n up to 6625109 (tested against the substitution method). But that is not enough, I don't want my device to crash after a few weeks. I use an Arduino XIAO, which has a SAMD21 processor, which has no 64 bit floating point (the only library is for ARM). But the formula can be rewritten, defining $a_n = (n+1)(\sqrt{2}-1) + \tfrac{1}{2}\sqrt{2}$ and $b_n = n(\sqrt{2}-1) + \tfrac{1}{2}\sqrt{2}$. Then $p(n) = \lfloor a_n \rfloor - \lfloor b_n \rfloor$ and the recursion equations are:

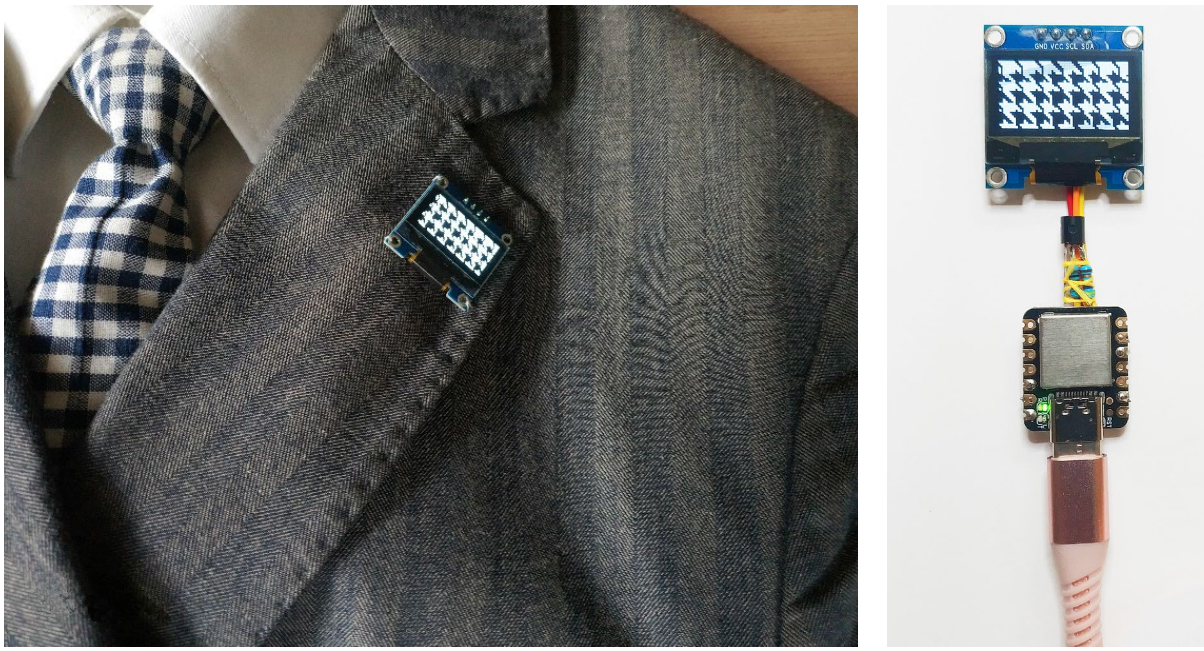
$$a_{n+1} = a_n + (\sqrt{2}-1),$$
$$b_{n+1} = a_n$$

with $a_0 = (\sqrt{2}-1) + \tfrac{1}{2}\sqrt{2} \approx 1.12132034355964257320$ and $b_0 = \tfrac{1}{2}\sqrt{2} \approx 0.70710678118654752440$. By approximation, $\sqrt{2}-1 \approx 0.41421356237309504880$. As one can see, only addition is needed. I implemented these equations in Arduino C++ code using three arrays of 21 bytes each: one for the digits of a, one for the digits of b, and one for the approximation of $\sqrt{2}-1$. Addition is done in a 21–step loop using a ripple carry approach. The floor function is easy: take the value at the first position of the array. In this way, the values of p_n for $n = 0,1,2,...$ appear one by one. Before the next iteration, the most significant digit must be cleared, preventing an overflow. This implementation works for n up to 10^{20}. If I show one new Pied-de-poule column per second, the program could work well for 10^{20}s, which is more than the estimated age of the universe: 13.7 billion years, that is 4.3×10^{17}s — although Penrose has said that there was another universe before ours.

Results

We show the working brooch in Figure 5. The LCD is the visible part of the brooch, whereas the Arduino microprocessor is hidden behind the lapel of the jacket. The four wires connecting the LCD and the Arduino are fed through the buttonhole of the jacket's lapel. A USB cable goes through a (newly sewn) buttonhole behind the lapel, so a power bank can be worn in the left inner pocket. The pattern scrolls slowly from right to left in steps of two pixels per second. A short video is available as supplementary material.

The brooch is powered from a 10000mAh power bank, commonly used as a back-up for smart phones. The brooch uses so little power that I had to add a pulsed transistor-resistor load circuit to keep the power bank awake. In this configuration, the brooch works for one week without charging.



(a) (b)

Figure 5: *Electronic brooch displaying the aperiodic Pied-de-poule tiling. (a) Overall look, OLED display: Seeed, shirt: Karl Lagerfeld, neck-tie: Laurentius.Lab., jacket: Exhibit®. (b) The brooch.*

Acknowledgements

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