Replacing Ammann's A4 Tiling Vertex Key by a Stitching Constraint

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Abstract

This paper presents a new way of presenting A4, the *fourth aperiodic set of tiles* by Robert Ammann, replacing the usual vertex constraint by a stitching constraint on the tiling. We demonstrate the equivalence between the two constraint systems and conclude by remarking the resulting stitching figure displays only Fibonacci Snowflakes.

Introduction

Given a set of two-dimensional tiles, the nature of the planar tilings that they admit arises from a deep interaction between the local and the global [7]. Aperiodic Tile Sets walk a fine line between order and disorder, admitting tilings, but only those without any translational symmetry, never permitting the simple repetition of periodic tilings. Constraints on the ways that pairs of tiles can be neighbours determine the structure of an infinite tiling, at all scales.

This paper focuses on Ammann's Fourth Aperiodic Set of Tiles, known as A4 [1] in a polyomino version and expresses a new way to embed local/global constraints on polyominoes borrowed from a traditional Japanese stitching technique known as "hitomezashi" [8].

Robert Ammann's Fourth Aperiodic Set of Tiles (A4)

The study of aperiodic sets of tiles started in the sixties, and reached global attention in the seventies when Roger Penrose issued his remarkable set of two tiles. An amateur mathematician, Robert Ammann, inspired by this discovery, found different sets of tiles (A2, A3, A4, A5)[6]. All these tilings present the unique characteristic of *keys*, meaning that a small fraction of the tiling embeds local constraints that have global effect.

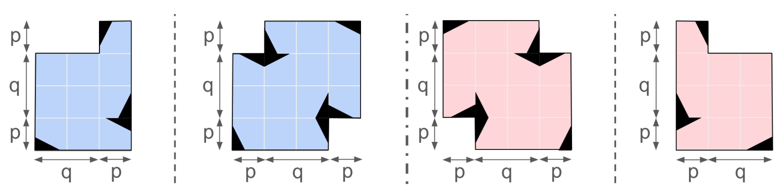

Figure 1: *Recto and verso of A4 ($p = 1$, $q = 2$) as set of marked polyominos.*

These keys can be represented in two different manners [1]: as a third tile encapsulating the key, or as a vertex decoration with a quarter of the key, which are *right triangles* — assuming that for any valid tiling, each surrounded marked vertex must be provided with a valid completed key. Figure 1 presents the vertex decoration version used in this work and an illustration of p and q dimensions. It is established [1] that any values for p and q dimensions are possible.

By using $p = 1$ and $q = 2$, we show in Figure 1 how to interpret A4 as marked polyominos. For tiling the plane, both sides of the tiles must be used. We will refer to them as *recto* and *verso*. In order to differentiate them in Figure 1, the recto [1, 6] is blue and the verso is red.

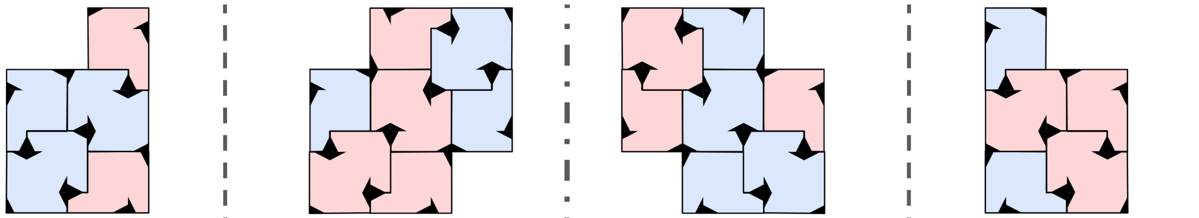

Figure 2: *Super-tiles for Figure 1 — rectos are identical to references [1, 6]*

In order to prove the existence of an aperiodic solution, it is common to present super-tiles — here given in Figure 2 for every tile of Figure 1. The proof that A4 is aperiodic [1] is performed by proving that any tiling starting with the large blue tile eventually came from a super-tile.

Replacing Ammann's Key by a Stitching Constraint

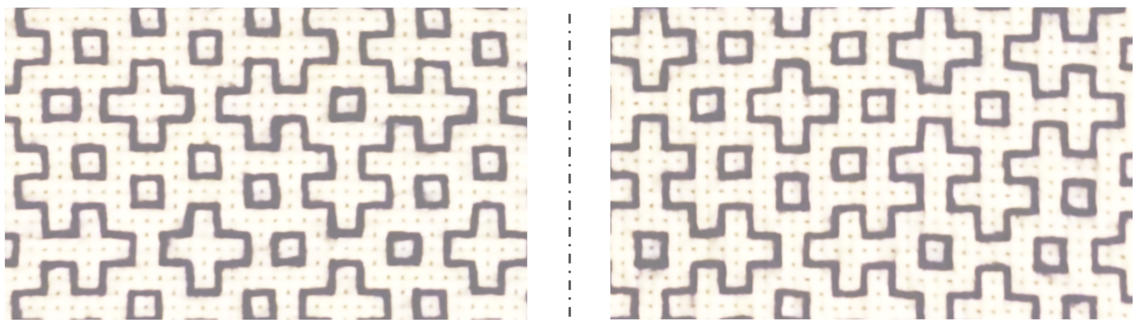

Figure 3: *Both sides of an Hitomezashi coaster, clipped from Figure 20 of [8].*

The Japanese word *sashiko* means *little stabs*, referring to the act of pushing one's needle vertically through thick layers of cloth[8]. In the contemporary world of global stitching practice there are basically four types of *sashiko*. Among them, *Hitomezashi* means *one-stitch sashiko*, and refers to *sashiko* that is completed by stitching regularly on a grid, one running stitch per grid spacing. Consider Figure 3 in which the threads that are above on the left side become invisible on the flip side of the coaster shown on the right, and vice-versa.

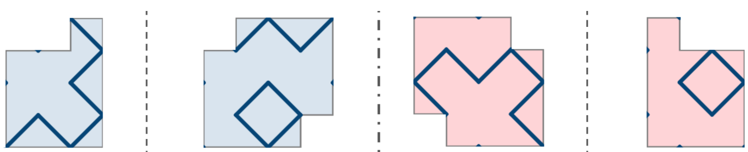

Figure 4: *Recto and verso of A4 ($p = 1$, $q = 2$) as set of hitomezashi polyominos.*

With this kind of constraint expression, we found a set of A4 tiles equipped with dashed lines which must be consistent on the whole plane in an hitomezashi manner. The grid used is oriented at 45° and has a step of $\sqrt{2}$! These new tiles equipped with our decorations are given in Figure 4. Note that the mirror tiles are precisely the mirror in hitomezashi meaning as coaster pieces with thread going on the alternate side of it. In the latter, we will call *A4-Classical* the Ammann's version with key tiles and *hitomezashi-A4* the version proposed in Figure 4.

Proposition 1. *Any A4-Classical solution is a hitomezashi-A4 solution.*

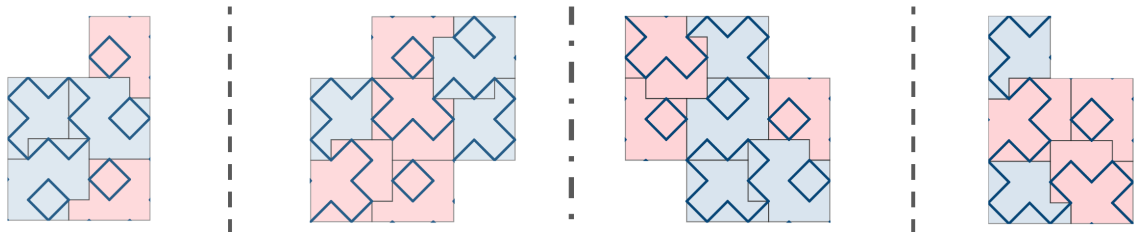

Figure 5: *Super-tiles for Figure 4*

Proof. Obtained by the provision of hitomezashi super-tiles in Figure 5. Assuming that (1) Any valid tiling in the classical form must be obtained by a super-tile substitution process [1] ; (2) Figure 5 can be superimposed to Figure 2 ; (3) Figure 7 proves the super-tiles to be resistant to recurrence ; we have: any A4-Classical solution can be translated in a A4-Classical substitution pattern, itself translated in a hitomezashi-A4 substitution pattern, providing an equivalent and valid hitomezashi-A4 solution. □

Proposition 2. *Any hitomezashi-A4 solution is an A4-Classical solution.*

Proof. In the other way, it is a bit more complex. As the hitomezashi constraints are a bit loose with respect to the strong vertex condition imposed by the classical key, the exploration of surroundings of a given tile has more elements than the equivalent demonstration in [1]. To manage that complexity, we developed a SAT engine as depicted in [7] except that our corona explorer computes single satisfying corona at a time with six rules and explores the resulting tree. The six rules are: (1) A SAT corona is valid if and only if it admits a *ring* of possible tiles around it that covers its *halo*[7] — its 8-neighbourhood minus itself — this condition precludes both holes and SAT coronas with no child; (2) Ancestor coronas halo cells must be used; (3) Previous sibling SAT coronas are excluded; (4) If a cell is used, then a ring tile containing it has to use it; (5) If a ring tile is used, then its cells are used; (6) Used ring tiles must be compliant two by two — both in non-overlap and hitomezashi constraints.

The exploration from the recto of the big tile should be non-terminating, but we adapt the proof of [1] and cut any branch that contains one of the three super-tiles given in Figure 6, adapted from that source. Due to the article format, please find more details in [4] where computation ends in less than two hours. It is a single HTML file running on a navigator, with a javascript translation of MiniSat[3] as oracle.

Therefore, any tiling of hitomezashi-A4 comes from a substitution pattern, and the same converse reasoning applies. □

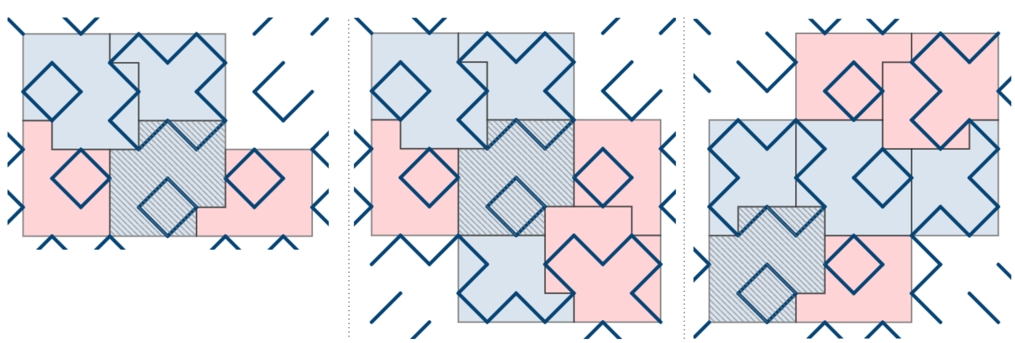

Figure 6: *Adaptation of [1] patterns which close exploration of super-tile matches.*

Conclusion

In this paper, we showed that a stitching constraint coming from Japanese traditions and brought to our sight by [8] can provide interesting prospects in the study of marked polyominos. To illustrate that potential, we did a little backtrack on our researches, to see where we diverged from A4. This backtracking effort results are in this paper, where we provide an *aperiodic set of tile with a stitching constraint* (hitomezashi-A4), and prove its equivalence with an *aperiodic set based on vertex constraint* (A4-Classical [1, 6]). One value of the replacement resides in the enhanced capacity to spot the flip side of the tile — in the original version, without the coloring, making a difference between the largest tile and its flip side in a random tiling takes some reasoning.

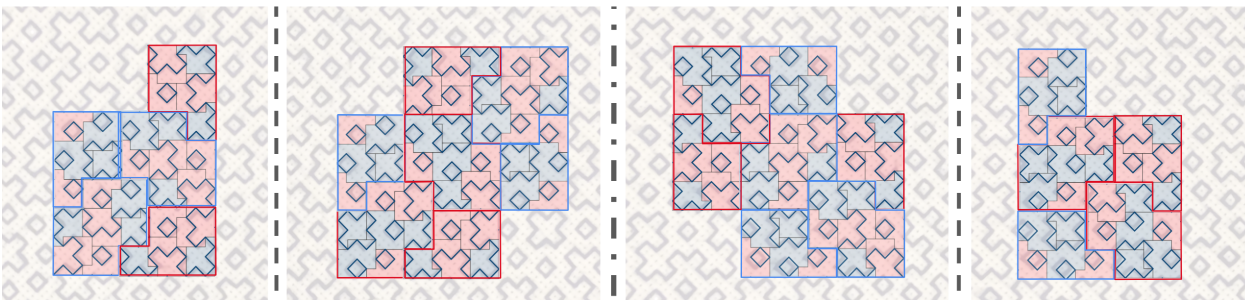

Figure 7: *Super-super-tiles superimposed with fragments of Figure 20 [8]*

There are several interesting side effects of this work. First, it opens a trail on a polyomino decoration system not extensively studied. Second, it highlights for a second time [5] a strong link between eight-fold Ammann's tilings (A4/A5) and the Fibonacci Snowflakes [2]. We conjecture that any *super*-*-tile produces an hitomezashi which can be superimposed on a fraction of a large enough iteration of Pell Persimmon [8] — conjectured there to host Fibonacci Snowflakes of same order. In Figure 7, we superimpose Super-super-tiles with fragments of the Fifth Pell Persimmon given in [8].

References

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