Hats in Grout: Practical Tiling with Aperiodic Monotiles

Georgia Crowther

Pittsburgh, Pennsylvania, USA; georgia.crowther@protonmail.com

Abstract

This paper explores the unique challenges of creating a traditionally laid tiled surface with aperiodic monotiles. The "hat" monotile will be used to demonstrate three main challenges in these practical tilings - maintaining consistent tile spacing for uniform grouting, implementing unintuitive, self-similar patterning on large surfaces, and manufacturing unusual tile shapes. Solutions to each of these challenges are presented.

Introduction

When physically laying a tiled surface - such as a floor, wall, or counter top - with familiar tile shapes and patterns, there are well-established and documented solutions to address common assembly challenges. However, these traditional methods break-down when applied to aperiodic monotiles (such as the "hat" and "spectre" tiles) [5] [6], thus presenting three unique challenges in the context of physically laid tiled surfaces - non-uniform grout spacing, unintuitive patterning, and difficulties in tile manufacturing. An example of each of these challenges with suggested solutions is presented using a sample tiling of a hat polykite with an edge-length ratio of 1:$\sqrt{3}$ (Figure 1).

Figure 1: *A functional sink backsplash made with "hat" tiles. Each tile is injection molded from recycled post-consumer plastic.*

Accommodating Grout

By definition, a tiling of the Euclidean plane must not contain gaps or overlaps between tiles [3]. However, physical tiling applications require a uniform gap between tiles to accommodate grouting. For many common planar tilings this gap can be easily and arbitrarily introduced (Figure 2a). We will show that this is not so for "hat" or "spectre" tiles (Figure 2b).

In traditionally tiled surfaces, properly applied grout is critical in making a long-lasting and finished-looking products. It is critical to maintain a consistent gap between tiles when applying grout, as different tile

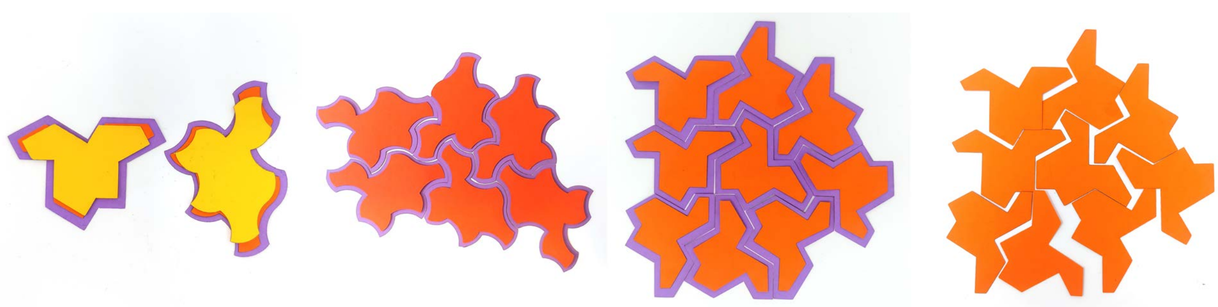
(a) (b)

Figure 2: *Regular hats cannot be tiled with uniform grout lines like regular polygons (a). Substantial deviations in grout width can be observed in (b).*

spacing may require different types of grout to prevent excessive grout shrinkage during curing and to cushion tiles against expansion and contraction [1]. Calculators for estimating the amount of grout and number of tiles required for an installation also assume uniform grout spacing.

For the purposes of this paper, and in the context of a physically laid tile, let us define an *effective tile* as the polygon formed by offsetting the edges of a tile by half of the width of a grout line. When physically tiling a surface, our concern is ultimately not whether our tiles admit tilings of the plane, but whether our *effective tiles* admit tilings of the plane.

All planar tilings will also tile the plane under any arbitrary affine transformation [3]. Therefore, if an *effective tile* is an affine transformation of its original tile, we will be able to tile the original tile with a uniform gap for grout. However, it is not generally true that uniformly offsetting the edges of a polygon is an affine transformation of that polygon.



(a) (b) (c)

Figure 3: *(a) A simple scaling (yellow) of a regular hat and spectre tile (purple) does not generate the same shape as a tile with offset edges (orange) (b) Using regular tiles as the* effective tile *and offsetting the edges generates our practical* hat-esque *and* spectre-esque *tiles (orange). (c) The* hat-esque *tile cannot tile the plane.*

A hat monotile cannot be tiled with a uniform grout spacing because a hat with uniformly offset edges is not an affine transformation of the hat (Figure 3a). Therefore, in order to tile a surface with hat (or ultimately, *hat-esque*) monotiles, we must start by setting the shape of our *effective tile* as the hat. We can then offset the edges of our *effective tile* by one half of our desired grout width (Figure 3b). Note that in this case, the shape of our tile and the width of our grout line cannot be altered independently. It is also important to note that the generated *hat-esque* tile cannot tile the plane itself (Figure 3c).

Non-Intuitive Patterning

When tiling, an adhesive substrate (typically a thinset or mastic) bonds tiles to the tiling surface. These substrates have a limited working time. Tiles must be placed within this working time or tile adhesion will be weak and prone to damage [1]. When working with periodic tilings, the location and orientation of a tile can generally be quickly deduced by observing the immediately adjacent tiles. However, with aperiodic tiles, arranging individual tiles is non-intuitive. It is generally not possible to correctly arrange a new tile solely by observing a few adjacent tiles. A single incorrectly oriented tile may appear at first to fit the aperiodic (but not random!) pattern, but it will soon result in a dead-end. Adjusting these incorrectly placed tiles wastes precious working time with your adhesive medium. It is therefore critical to have a method in place for efficiently and correctly placing hat tiles on your surface. The advice for efficiently and correctly placing hat monotiles can be reduced to two suggestions - Operate in metatiles, and prepare patches using the *indirect method*.

Figure 4: *Metatiles are used to generate supertiles (which can be used to make larger supertiles) before thinset is spread.*

Working in the metatiles (or supertiles) as proposed in [5] greatly simplifies the complexity of larger aperiodic tilings. Prepare metatiles and supertiles one to two levels below the scale of your final tiling as shown in Figure 4. Start by preparing the smallest possible metatiles and building-up supertiles until you can simply pattern supertiles to complete your tiling. Choosing different tile colors for metatiles, such as those suggested in Figure 2.1 of [5] and as shown in Figure 4, will help your orient yourself as you generate your metatiles and final tilings.

To make aperiodic tiling even more efficient, the *indirect method* of preparing your tiles, as described in [4], is recommended. Pattern your metatiles, with your grouting gap, on a large, flat surface. Lower a piece of contact paper onto your completed pattern and press your contact paper firmly against your tiles. The paper and attached tiles may then be lifted as a unit and placed on your tile adhesive. Use a stiff flat surface, such as a section of plywood, and a mallet to firmly press the tiles into your adhesive. Once the tile adhesive has set, you may remove the contact paper. This method may be used to either pre-tile your whole surface or to pattern large supertiles.

Tile Production

Unlike most traditional tile shapes, *hat-esque* and *spectre-esque* tiles are both non-convex and do not tile the plane. It is very difficult to create interior angles from materials that must be cut or chipped into shape, such as glass or stone, limiting their use for practical aperiodic tilings. For materials that can be cut with a water-jet or laser-cutter, *hat-esque* and *spectre-esque* tiles will leave substantial waste material (as shown in Figure 3c). Therefore, in order to produce reasonable numbers of tiles with minimal waste we are limited to manufacturing aperiodic monotiles from materials that can be molded, cast, or extruded - such as ceramics, porcelain, or, as in this case, plastics.

A simple mold can be used to produce *hat-esque* tiles quickly and with practically no waste. In the case of the backsplash shown in Figure 1, recycled plastics (chiefly polypropylene) were molded into the mold shown in Figure 5 using a Morgan-Press G-100T model injection molding machine. Further detail of this process is documented at [2]. A similar method could be used with a number of other materials. A critical advantage of injection molded recycled plastics is that they can be made relatively quickly and that both sides of the tile are usable (to simplify the manufacture of mirrored hats).

Figure 5: *The mold used to produce injection molded hat tiles from recycled plastic.*

It is generally recommended that you procure 8-10% extra tiles for your project to account for edges and waste [1]. However, I recommend increasing that to 12-15% for hat tiles (and mirrored hats, if your tiles cannot be flipped) since you will not be able to exploit tile symmetry to reduce waste at borders.

Summary and Conclusions

Though physical tilings with aperiodic monotiles like the "hat" present unique challenges, these can be easily overcome with some patient planning before laying your thinset. The additional work in adjusting tile shape for uniform grout spacing, pre-placing your supertiles, and efficiently producing tile is well worth the effort for the eye-catching, unique, and mathematically beautiful finished product.

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