3D Weaving Patterns and Modular Kirigami

Mircea Draghicescu

Portland, Oregon, USA; mircead3@gmail.com.com

Abstract

We extend the concept of weaving from the plane to the three dimensional space, describe several infinite weaving patterns and illustrate them with many models. We also describe modular kirigami techniques that can be used to build interesting 3D structures based on these patterns.

Introduction

In weaving, threads parallel to two or more axes in the plane are interlaced to form a fabric. Two such weaving patterns are known: the *biaxial* (or *plain weave*) pattern, with two axes and the *triaxial* (or *hexagonal*, *kagome*) pattern with three axes, see Figure 1 (a),(b). Different designs can be obtained by changing the way in which the threads cross each other; in the simplest case, crossings alternate in an over/under manner.

There are many ways to extend the concept of weaving from 2D to 3D. A flat weave can be bent to make baskets and other three-dimensional objects. More generally, [1] shows how the kagome pattern can be used to weave complex, three-dimensional surfaces, such as the ones showcased in [9]. Weaving polyhedra surfaces that are topological spheres is explored in [4] and [5]. The surface of a polyhedron whose vertices have degree 4 can be woven with threads that follow the edges of the polyhedron; the threads can always be arranged to cross each other alternately over/under. In [2] we examine a class of polyhedra that can be woven in this way with a single, self-intersecting, loop. Related to 3D weaving is the crystallography topic of cylinder (or rod) packing, see, for example, [10], [11]. Symmetric 3D weaves of helices are described in [14]; two examples are shown in [13]. A technique for generating 3D weaves by projecting line arrays in the hyperbolic plane onto triply periodic minimal surfaces is described in [3].

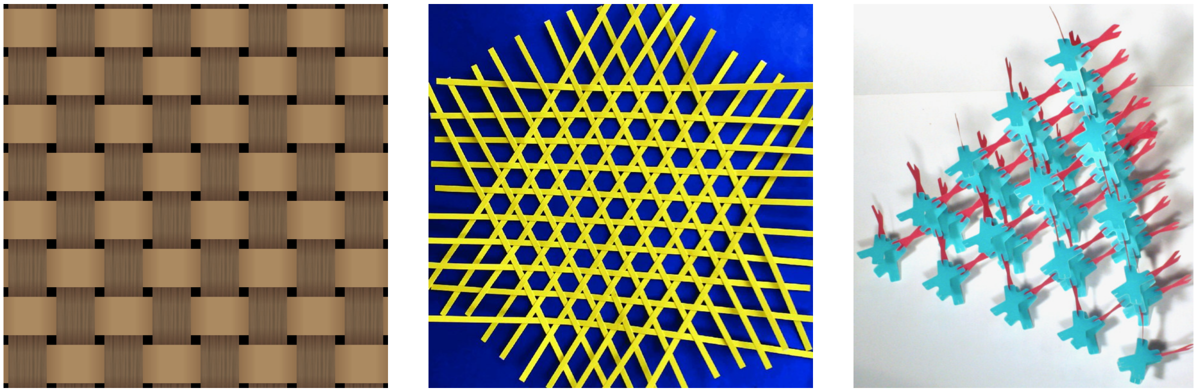
New, evolving, manufacturing methods can stitch together several layers of fabric into a “thick” weave by adding vertical strands to the warp and weft strands of the 2D weaving. These technologies have engineering applications that extend beyond the textile industry, see, for example, [7], [8], [12].

We will define 3D weaving here in a way that directly generalizes flat weaving. While the lines abstracting the threads of the two 2D weaving patterns can be generated by translating a (flat) polygon, the lines of a 3D weaving pattern are generated by translating a skew polygon (a polygon whose vertices do not lie in a plane). Threads following these lines can be interlaced, over/under, in different planes and intersecting planes are connected by their common thread. The result is a fully connected, conceptually space-filling, weave.

We illustrate the 3D weaving patterns with various models, including paper models built using modular kirigami methods. In one of these methods, threads are replaced with same shape pieces and a weaving pattern becomes a template for building complex, modular 3D structures, see Figure 1 (c).

Weaving Patterns

Informally, a weaving pattern is an infinite set of lines that are parallel to a (finite) number of axes and intersect each other in a regular manner. A weaving pattern will be defined with reference to a generating polygon. Not all polygons generate weaving patterns; we will impose a condition that ensures that only two lines meet at each intersection point and no pattern lines meet at some “extraneous” intersection point.



(a) *2D biaxial* (b) *2D triaxial, image from [1]* (c) *Modular kirigami model based on a 3D four-axial pattern*

Figure 1: *Weaving patterns.*

Formally, let P be a skew polygon such that any two parallel edges, if they exist, have the same length and let $k > 1$. For an edge e of P denote by $|e|$ the length of e and by $\bar{e}$ the line extending e. A translation along $\bar{e}$, in either direction, by distance $k\,|e|$ will be called a *pattern translation*; let $L(P, k)$ be the closure, under pattern translations, of the set of lines $\bar{e}$ for all edges e, i.e., the set of lines $t(\bar{e})$ for all edges e and all sequences of pattern translations t.

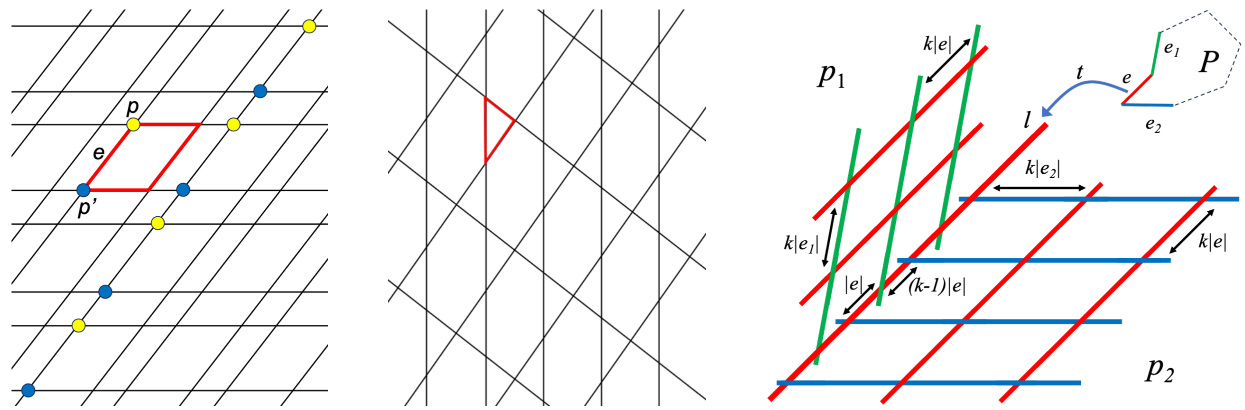
If e, e' are two edges of P adjacent to vertex v and t is some sequence of pattern translations, then the $L(P, k)$ lines $t(\bar{e}), t(\bar{e'})$ intersect in $t(v)$. Conversely, if two lines $t(\bar{e}), t'(\bar{e'})$ intersect only if e, e' are adjacent, then $L(P, k)$ is the *weaving pattern* (or just *pattern*) generated by P, k, otherwise P, k does not generate a pattern.

The weaving pattern is 2D if P is planar and 3D otherwise. Figures 2 (a) and (b) show the 2D patterns generated by a parallelogram and a triangle, respectively. The edges of P, choosing only one edge in a set of parallel edges, specify the axes of the pattern. The pattern is symmetric with respect to the pattern translations; any symmetries of P also induce corresponding symmetries of the pattern.

The set of intersection points of the lines in a weaving pattern $L(P, k)$ is $V(P, k)$, the closure of the set of vertices of P under pattern translations. Note that only two pattern lines intersect in each point in $V(P, k)$. Referring to Figure 2 (a), if e is an edge of P with endpoints p, p', a pattern line obtained by pattern translations from $\bar{e}$ contains the intersection points obtained by translating p, equally spaced by $k\,|e|$ (yellow), interspersed with the intersection points obtained by translating p', also equally spaced by $k\,|e|$ (blue); the two sets of points are shifted by $|e|$ with respect to each other. The special value $k = 2$ makes all the intersection points along a pattern line equally spaced, see Figure 2 (b). This value of k simplifies the model construction and is used in the models of the patterns described in the 3D Pattern Examples section.

For a skew polygon P, denote by $C(P)$ the condition that the extensions of any two non-consecutive edges of P do not intersect. From the pattern definition, if P, k generates a pattern then $C(P)$ is true. In 2D, $C(P)$ is satisfied only by parallelograms and triangles, which generate, for any $k > 1$, biaxial and triaxial patterns, respectively. In 3D, $C(P)$ is necessary, but not sufficient to ensure that P generates a pattern for all k. We conjecture that for any skew polygon P, if $C(P)$ then there exists a $k > 1$ such that P, k generates a weaving pattern.

Intuitively, for any skew polygon P and $k, k' > 1$, the patterns $L(P, k)$ and $L(P, k')$, if they exists, are, essentially, the same. Moreover, $L(P, k)$ is essentially the same with $L(P', k)$ where P' is obtained from P by



(a) *2D biaxial generation, $k = 1.5$* **(b)** *2D triaxial generation, $k = 2$* **(c)** *Interlaced threads in a 3D pattern*

Figure 2: *Weaving patterns diagrams.*

changing its edge lengths, or, with some restrictions, its angles. More precisely, we will say that two patterns are *similar* if there is a one-to-one correspondence between the edges of their generating polygons which preserves the relations of intersection and parallelism. The 2D biaxial and, respectively, triaxial patterns are similar; as another example, all the 3D patterns generated by skew quadrilaterals are similar.

Interlacing the 3D Patterns

We will now show how a 3D weaving pattern can generate a 3D weave where threads following the pattern lines cross each other in an over/under manner. Consider the 3D pattern generated by a skew polygon P and some $k > 1$. A plane defined by two intersecting lines in $L(P, k)$ will be called a *weaving plane*. Any weaving plane is parallel to some plane (e, e') defined by two consecutive edges e, e' of P. The pattern lines in this plane are parallel to either e or e' and the threads following these lines can be interlaced over/under.

Each pattern line is contained in either one or two weaving planes; when the line is the intersection of two weaving planes, the thread on this line is interlaced, separately, in each plane thus connecting the weaves on the two planes. In the end, all threads are interlaced producing a fully connected, space-filling weave. More precisely, let e_1, e, e_2 be three consecutive edges in P. Figure 2 (c) shows a fragment of the pattern with lines parallel to these edges colored with the respective edge color (lines parallel to e_1 are green, etc.) There are two cases:

1. e_1 and e_2 are not parallel. e_1 and e_2 cannot be co-planar, otherwise P does not generate a weaving pattern (it does not satisfy $C(P)$), so $(e_1, e) \neq (e, e_2)$. Referring to Figure 2 (c), consider the red pattern line l obtained by some sequence of pattern translations t from $\bar{e}$. The weaving plane $p_1 = t(e_1, e)$ contains two families of parallel, equally spaced lines: green, spaced by $k\,|e|$, and red, spaced by $k\,|e_1|$; threads running along these lines can be interlaced. Similarly, the weaving plane $p_2 = t(e, e_1)$ contains blue and red lines and threads running along these lines can be interlaced. The crossing points on $l = p_1 \cap p_2$ that are are used for interlacing in p_1 are interspersed with the crossing points used for interlacing in p_2; l thus connects the weaves on p_1 and p_2.
2. e_1 and e_2 are parallel and thus $(e_1, e) = (e, e_2)$. In Figure 2 (c), imagine planes p_1 and p_2 folded into a single plane p with the red lines superimposed and the green and blue lines, which are now parallel, interspersed. In p, the green and blue threads (spaced, alternately, by $|e|$ and $(k-1)\,|e|$) can be interlaced with the red threads spaced by $k\,|e_1| = k\,|e_2|$. The red lines in p are not in any other weaving plane; p

will be connected to other weaving planes via green and blue threads, as described in case 1. This is always possible since none of the other two edges adjacent to e_1 and e_2 can be parallel to e, otherwise either P is a parallelogram (and thus the pattern is not 3D), or P does not satisfy $C(P)$ and thus does not generate a pattern.

We will call *p dense* since it contains twice as many lines parallel to e_1 and e_2 (both green and blue) as in a "normal", non-dense weaving plane such as p_1 and p_2 in case 1.

Modular Kirigami

Kirigami is the Japanese art of cutting and folding paper into three-dimensional objects or designs. *Modular kirigami*, introduced in [6], refers to the art of assembling complex structures out of cut paper modules.

3D interlacing, as defined in the previous section, does not work well when the strands are flat strips of material e.g., paper strips. Normally, the strips are contained in the weaving plane, so a strip connecting two weaving planes would have to be in two different planes. To overcome this issue, we will interlace the paper strips in a different way using two methods inspired by modular kirigami.

The first technique consists in making paper strips that can "pierce" each other by cutting equally spaced slits along the middle of the strips so that intersecting strips can be inserted through them. The new interlacing method consists in a thread (paper strip) alternately piercing and being pierced by the intersecting threads; models built in this way are shown in Figures 3 (d) and 6 (b). The slits are centered on every other intersection point of the respective pattern line and the length of a slit is at least equal to the width of the strip and might be longer to accommodate strips intersecting at acute angles. This method produces more resilient weaves since a thread cannot be pulled out, but also complicates the weaving process.

For the second method, imagine that the strips of paper with slits are cut into pieces on lines bisecting each slit. The half-slits thus become notches that can be used to latch a piece onto an intersecting piece. In practice, in order to strengthen the connections, these notches are matched with new notches on the intersecting piece; each piece thus has four notches. Each piece lies in a weaving plane and corresponds to a pattern intersection point with the notches marking the lines through this point. This method makes the construction process fully modular: the size of models is not limited anymore by the length of the strips and arbitrary large models can be build by connecting more and more pieces, see Figure 1 (c). We used playing cards as modules for several models, see Figures 3 (e) and 6 (c). Although the connections via notches work better when the pieces are not in the same plane, this method can be used even for 3D patterns with dense weaving planes (case 2 in the previous section) which require connecting co-planar modules, see Figure 4 (c) and Figure 5 (c).

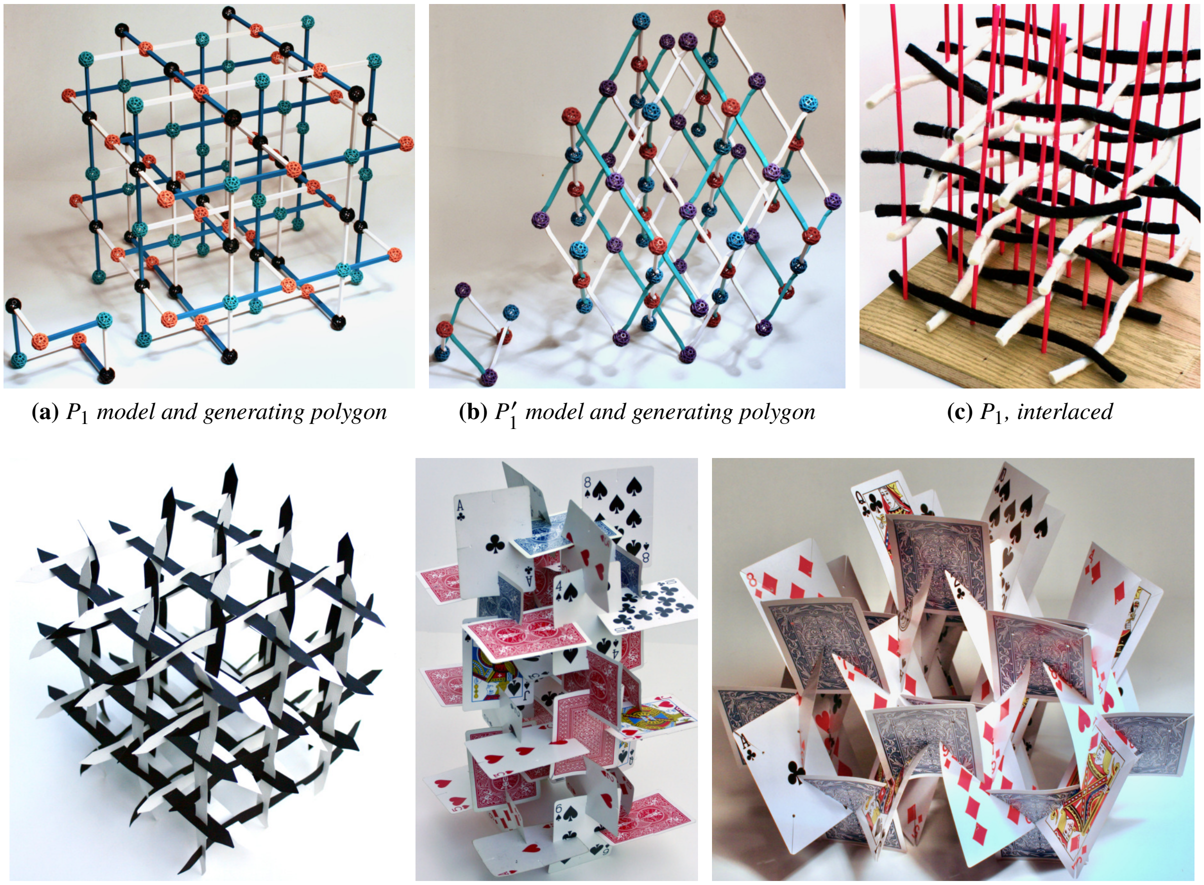
3D Pattern Examples

The four simple patterns described below are generated by skew polygons with congruent sides and vertices that are a subset of the vertices of the unit cube. Since we have chosen $k = 2$, their intersection points are on the 3D grid of points with integer coordinates.

In the interlaced models (e.g., Figure 3 (c)), the threads are colored according to their parallel axis, while in the models built with paper strips (e.g., Figure 3 (d)) we used just two colors such that intersecting threads do not have the same color; this is possible since all four generating polygons have an even number of sides.

The first three patterns, P_1, P_2, and P_3, are triaxial with the X, Y, and Z axes of the coordinate system as axes; their lines are thus some of the integer grid lines parallel to the axes. They all have three pattern translations, with distance 2, along the X, Y, and Z axes.

The pattern $P_1 = L(H_1)$ is generated by the regular skew hexagon H_1 with vertices, in order, $(0, 0, 1)$, $(1, 0, 1)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, $(0, 1, 1)$, see Figure 3 (a). It is easy to prove that $L(H_1)$ is the set of lines



(a) *P_1 model and generating polygon* **(b)** *P_1' model and generating polygon* **(c)** *P_1, interlaced*

(d) *P_1, paper strips with slits* **(e)** *P_1, modular* **(f)** *P_1', modular*

Figure 3: *Weaving patterns P_1 and P_1'.*

$(x = m, y = n)$, $(y = m, z = n)$, $(z = m, x = n)$, for all m, n such that $m + n$ is odd, and that H_1 indeed generates a weaving pattern.

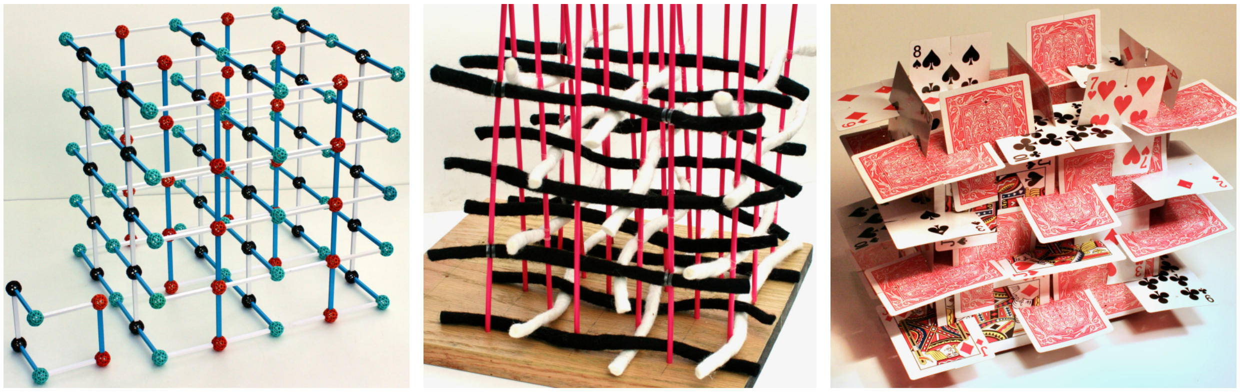
$V(H_1)$ is the set of points (a, b, c) such that a, b, c are not all even or all odd. In fact, another way to describe P_1 is as the set of all integer grid lines parallel to the axes which do not pass through any grid point whose coordinates are all even or all odd.

The weaving planes are parallel to the XY, YZ, and ZX planes, intersecting the axes at every integer point. None of them are dense and the lines in each plane form a square grid of size 2; the grids on adjacent parallel planes are offset by 1 in both directions. P_1 has all the symmetries of the cubic lattice of unit 2.

The model in Figure 3 (c) is built with 18 red, vertical sticks (two 3×3, size 2 grids, shifted with respect to each other by 1 in both directions) and 12 each white and black thick yarn threads. Each red thread is interwoven with 2 white and 2 black ones in two perpendicular planes and each white (black) thread is interwoven with 3 red and 3 black (white) ones in two perpendicular planes.

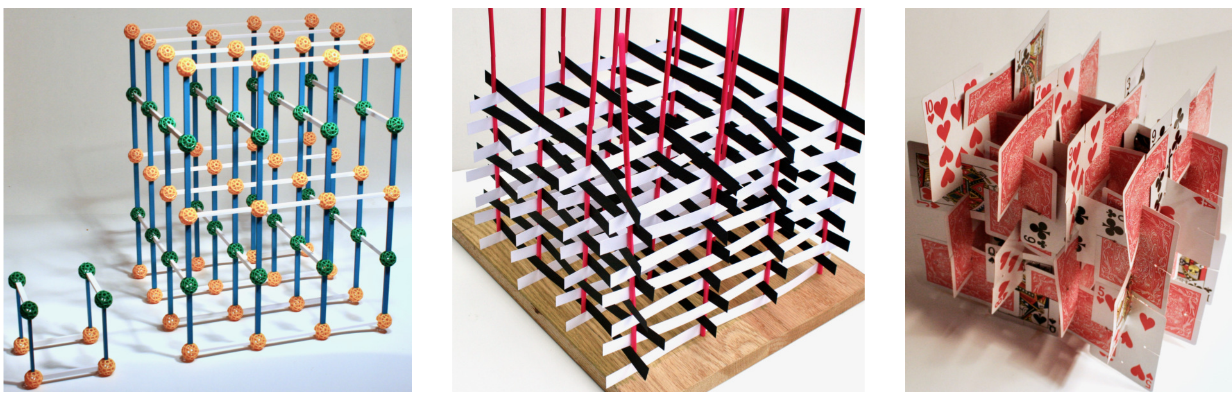
By changing the angles of H_1 we obtain the similar pattern P_1' (Figure 3 (b)) which has a different-looking modular model (in Figure 3, compare (e) with (f)).

The pattern $P_2 = L(H_2)$ is generated by the skew hexagon H_2 with vertices $(0, 0, 0)$, $(0, 1, 0)$, $(0, 1, 1)$, $(0, 0, 1)$, $(1, 0, 1)$, $(1, 0, 0)$, see Figure 4 (a); note that P_1 and P_2 are not similar. The lines in $L(H_2)$ are $(x = 2m, z = n)$, $(y = 2m, z = n)$, for all integers m, n and $(x = m, y = n)$ for all m, n such that $m + n$ is odd.



(a) *Zometool model and generating polygon* **(b)** *Interlaced* **(c)** *Modular*

Figure 4: *Weaving pattern P_2.*



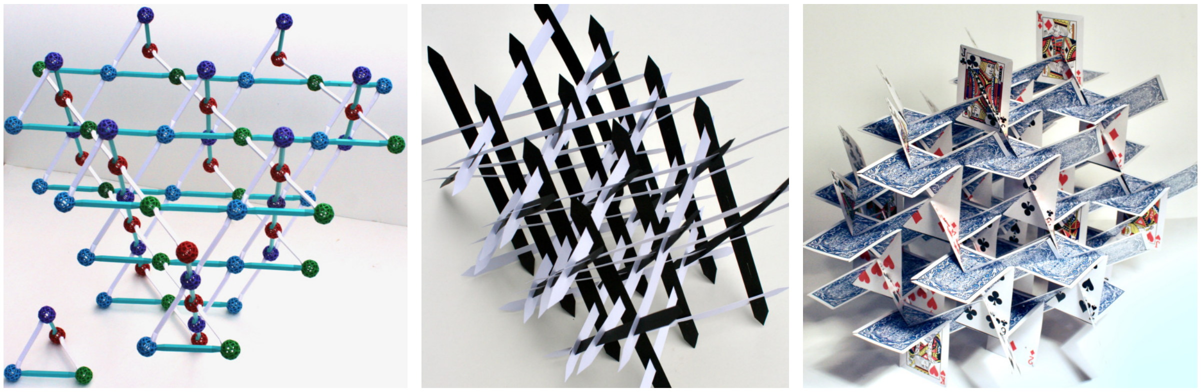
(a) *Zometool model and generating polygon* **(b)** *Interlaced* **(c)** *Modular*

Figure 5: *Weaving pattern P_3.*

$V(H_2)$ contains all points (i, j, k) such that at least one of i, j is even. There are three families of parallel weaving planes: $(x = 2m)$, $(y = 2m)$, and $(z = m)$, for all integers m. The $(z = m)$ weaving planes contain a square grid of size 2. The $(x = 2m)$ and $(y = 2m)$ weaving planes are dense, with lines parallel to Y, respectively X, spaced at a distance of 1 and lines parallel to Z spaced at 2. The model in Figure 4 (b) has the same number of threads as the one in Figure 3 (c). The vertical red sticks are in the same configuration, but the white and black threads are now aligned in their respective (dense) weaving planes

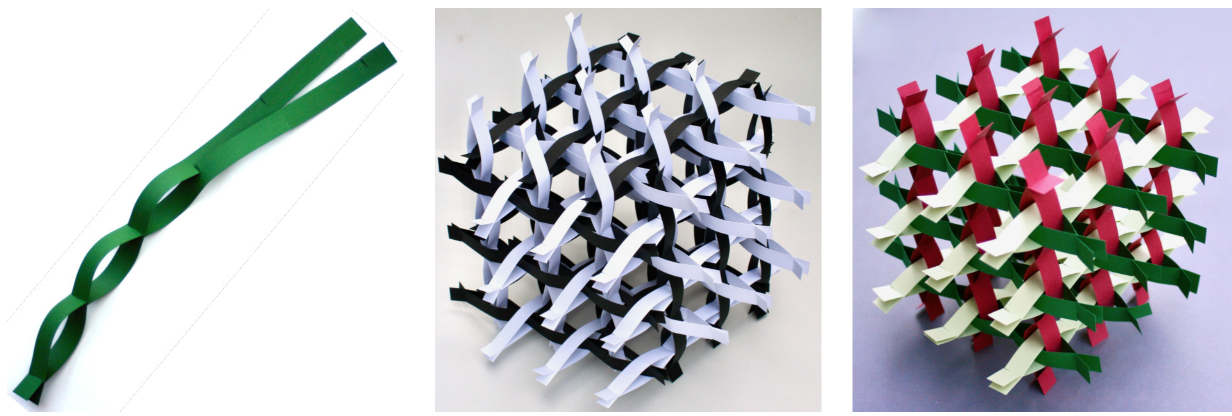
The pattern $P_3 = L(O)$ is generated by the skew octagon O with vertices $(0,0,0)$, $(0,0,1)$, $(0,1,1)$, $(0,1,0)$, $(1,1,0)$, $(1,1,1), (1,0,1), (1,0,0)$, see Figure 5 (a). $L(O)$ lines are $(x = m, y = n)$, $(x = m, z = 2n + 1)$, and $(y = m, z = 2n)$ for all integers m, n. $V(O)$ contains all integer grid points. There are two families of parallel, dense weaving planes: $(x = m)$ and $(y = m)$, for all integers m. A weaving plane contains lines parallel to Z spaced at 1 and parallel to Y, respectively X, spaced at 2. The model in Figure 4 (b) has 16 red vertical sticks in a 4×4 grid and 28 each white and black paper threads. We could use paper strips for the threads parallel to X and Y since each such thread lies in a single weaving plane.

The four-axial pattern P_4 is generated by the regular skew quadrilateral Q with vertices $(0,0,0)$, $(1,0,1)$, $(1,1,0)$, $(0,1,1)$. The axes are the four diagonals of the XZ and YZ planes. $L(Q)$ has four families of



(a) *Zometool model and generating polygon* **(b)** *Paper strips with slits* **(c)** *Modular*

Figure 6: *Weaving pattern P_4.*



(a) *Intertwined strips.* **(b)** *P_1 model built with interlaced strips.* **(c)** *Weaving the full cubic lattice.*

Figure 7: *Another kirigami technique.*

parallel lines: $L_1 = (x = 2m, y - z = 2n)$, $L_2 = (y = 2m, x - z = 2n)$, $L_3 = (x = 2m + 1, y + z = 2n + 1)$, $L_4 = (y = 2m + 1, x + z = 2n + 1)$, for all m, n. There are four families of parallel weaving planes: $P_{12} = (x + y - z = 2m)$, $P_{23} = (-x + y + z = 2m)$, $P_{34} = (x + y + z = 2m)$, and $P_{41} = (x - y + z = 2m)$, for all m. A plane in P_{12} contains some lines in L_1 intersecting with some lines in L_2, a plane in P_{23} contains lines in L_2 and L_3 etc. A line in L_1 does not intersect any line in L_3, same for L_2 and L_4. $V(Q)$ is the set of points (i, j, k) such that $i + j + k$ is even.

Another Kirigami Method

Another kirigami technique which can be used for 3D weaving consists in building each thread out of two paper strips with equally spaced notches cut alternately on each side of the strip, see Figure 7 (a). Intertwining the strips produces a thread with "holes" through which the intersecting threads can go through; Figure 7 (b) shows a P_1 model built with this kind of threads.

This method can even accommodate intersections of three threads. Figure 7 (c) shows a model of the full cubic lattice, which is *not* a weaving pattern as defined above. Note how three threads pass through each lattice point, enclosing each other in a circular, "Borromean rings"-like manner: green encloses red, red encloses yellow, yellow encloses green.

Conclusions and Future Work

We extended the concept of infinite weaving from two to three dimensional space by introducing the concept of a weaving pattern. We described a few weaving patterns (there are many more) and illustrated them with models built using several methods. We showed how to use kirigami techniques to build modular, space-filling sculptures based on these patterns and plan to create some other ones based on these ideas.

In the future, we would like to investigate the connections between the 3D weaving patterns and other space-filling structures such as infinite polyhedra and honeycombs.

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