Attainable Symmetry in Generalized Hitomezashi Patterns

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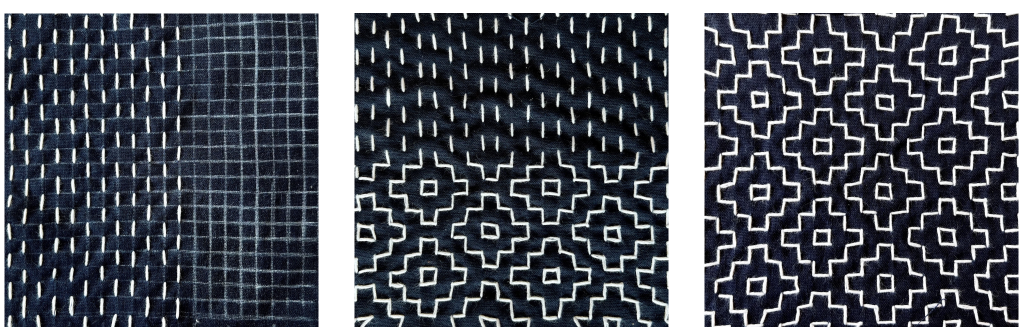
Abstract

Hitomezashi sashiko is garnering increased attention in the mathematics community. This Japanese-style of embroidery is as beautiful as it is mathematical—certain hitomezashi patterns can be encoded by two binary strings. In this paper, we define generalized hitomezashi patterns as those that can be generated by two binary strings. We provide properties on the generating binary strings that will produce symmetries in the resulting design and use these properties to prove which wallpaper groups are possible symmetry groups for periodic generalized hitomezashi patterns.

Introduction

Sashiko is an embroidery practice that originates in Japan during the Edo era (1615-1868). Its name literally translates to "little stab," which is an apt characterization of the small stitches used in this embroidery art. Traditionally, sashiko was used to mend damaged fabric by sandwiching layers of fabric together and using sashiko stitching to reinforce the layers. In its modern form, sashiko has found its way into the work of not just those looking to mend fabric, but for those wanting to impart all-over embroidery designs on fabric or quilts. For those interested in reading more about the history, evolution, and techniques of sashiko, we point the reader to the wonderful book of Susan Briscoe [1].

While there are a few different styles of sashiko, this paper focuses on the hitomezashi style (one stitch sashiko). Hitomezashi is worked on a two-dimensional grid of horizontal and vertical lines. We examine a particular type of hitomezashi pattern: stitching is done on a square grid of rows and columns, the length of each stitch is the length of the side of a square, and along any straight line of stitching the thread alternates above and below the fabric. Whether you start a row or column of stitching above or below the fabric determines the entire line of stitches. The steps of an in-progress hitomezashi are given in Figure 1.



(a) (b) (c)

Figure 1: *The process of stitching the Kakinohanazashi, or persimmon flower, stitch is shown. Note that vertical columns of stitches are stitched separately from horizontal rows of stitches.*

These kinds of hitomezashi patterns can be encoded using two binary strings, where a "0" (resp. "1") means stitching starts below (resp. above) the fabric. One string lies along the bottom of the design and determines the vertical rows of stitches; the other lies along the left of the design and determines the horizontal rows of stitches—an example of this correspondence is provided in Figure 2a.

In this paper, we are expanding beyond traditional hitomezashi patterns to those that are generated with any binary string we choose. To recognize this fact, we call the patterns we generate *generalized hitomezashi patterns* (GHPs for short).

The mathematics of hitomezashi was brought to the attention of the Bridges community in 2020 by Hayes and Seaton [4]. Since then, hitomezashi has appeared in many papers (for example, Defant and Kravitz address loops in hitomezashi in [2]); a quick search of ArXiv shows that mathematics done pertaining to sashiko and hitomezashi is occurring at an increasing rate. However, we have not seen proofs detailing the attainable symmetries in GHPs. Mathematical "symmetry samplers" have been done in a number of different crafts; this practice is well-documented by Goldstine [3]. We follow this practice by proving which wallpaper symmetries are attainable under this particular framework of hitomezashi.

Since we can encode our GHPs with binary strings, we can find properties on the generating binary strings that will imply particular symmetries of the design. These properties are natural and easy to check, so we first introduce how those properties are determined. From there, we prove that there are only nine attainable wallpaper symmetry groups in GHPs: p1, p2, pm, pg, cm, pmm, pmg, cmm, and p4m. We do not consider the possible Frieze symmetries that can be obtained by GHPs because all 7 Frieze patterns are attainable.

Encoding Patterns with Binary Strings

We define two binary strings $x = x_1 \dots x_n$ and $y = y_1 \dots y_m$, where x is placed along the bottom and defines the vertical stitches and y is placed along the left and defines the horizontal stitches; both words start in the lower left-hand corner. This yields a design that measures $n - 1$ by $m - 1$. For example, the design generated by $(x, y) = (11001011, 0101011)$ is pictured in Figure 2a.

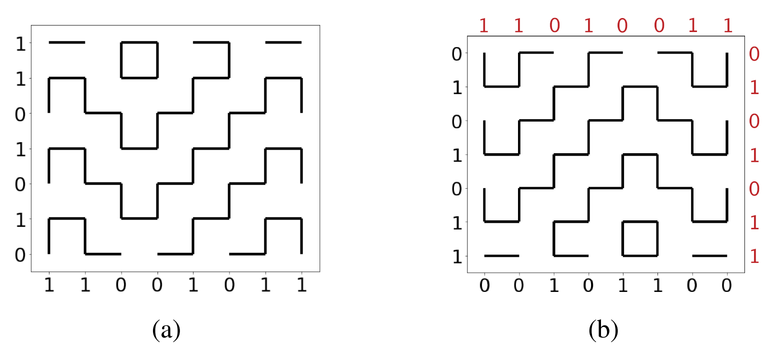


Figure 2: *Figure (a) shows the GHP generated by* $(11001011, 0101011)$. *Figure (b) shows the same design rotated by* $180°$ *and shows* $x' = 11010011$ *and* $y' = 1101010$ *written in red.*

Encoding GHPs with binary strings means we can understand the symmetries of a design directly from the binary strings. To this end, we work to characterize when certain symmetries are implied by properties on the binary strings. Before we proceed, we need a few definitions.

Definition 1. Given a binary string $a = a_1 \dots a_n \in \{0,1\}^n$ we can define the *reverse* as $a^R = a_n a_{n-1} \dots a_1$ and the *complement* as $a^C = (1 - a_1) \dots (1 - a_n)$. Additionally, the *reverse-complement* is $a^{RC} = (1 - a_n)(1 - a_{n-1}) \dots (1 - a_1)$.

For example, when $a = 1010001$, we have $a^R = 1000101$, $a^C = 0101110$, and $a^{RC} = 0111010$.

As we move forward, the parity of n and m becomes an essential notion. For any bounded GHP, it is possible to associate a binary string to the top and right sides of the design: we will call these words x' and y' respectively (x' starts in the top left and y' starts in the bottom right of the design). Then we find that

$$x' = \begin{cases} x & \text{if } m \text{ is even} \\ x^c & \text{if } m \text{ is odd} \end{cases} \qquad \text{and} \qquad y' = \begin{cases} y & \text{if } n \text{ is even} \\ y^c & \text{if } n \text{ is odd.} \end{cases}$$

These results arise from the fact that if y has length m, then there are $m-1$ vertical stitches along each column. So the vertical stitches will end on the different side of the fabric if m is odd and on the same side of the fabric is m is even (and similarly for the horizontal stitches). We can observe this in Figure 2b. The vertical stitches that start on the top of the fabric, end on the bottom and vice versa since $m = 7$ is odd; so $x' = x^C = 11010011$. However, the horizontal stitches begin and end on the same side of the fabric since $n = 8$ is even and we get $y' = y = 1101010$.

Symmetries of Generalized Hitomezashi Patterns

In this section, we define how different reflections and rotations affect the binary strings of our GHPs. With this knowledge, we can then recognize whether a design has a reflective or rotational symmetry only from the binary strings. Since we are working on a grid of squares, we only need to work with reflections and rotations that are associated with the symmetries of a square, D_4. These are the counterclockwise rotations of degrees 0°, 90°, 180°, and 270° notated as e, r_{90}, r_{180}, and r_{270}, respectively; and reflections R_H, R_V, R_D, and R_A as pictured in Figure 3. Note that we will only consider R_D and R_A when $n = m$, but otherwise do not require a square design.

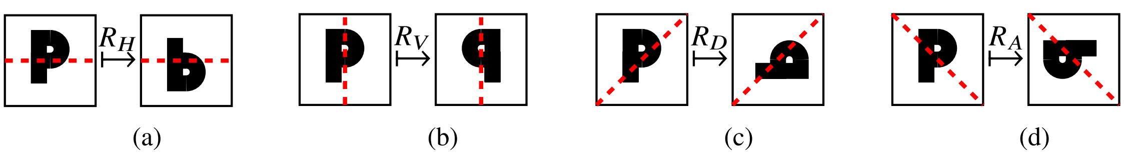


Figure 3: *The four different reflections in D_4.*

Lemma 2. *The reflections R_H, R_V, R_D, and R_A have the following effects on the binary strings that generate a GHP:*

$$R_H(x, y) = \begin{cases} (x, y^R) & m \text{ even} \\ (x^C, y^R) & m \text{ odd} \end{cases} \qquad R_V(x, y) = \begin{cases} (x^R, y) & n \text{ even} \\ (x^R, y^C) & n \text{ odd} \end{cases}$$

$$R_D(x, y) = (y, x) \qquad R_A(x, y) = \begin{cases} (y^R, x^R) & n \text{ even} \\ (y^{RC}, x^{RC}) & n \text{ odd.} \end{cases}$$

Proof. For R_H, the binary word, x', along the top edge becomes x in the resulting design, and y is clearly reversed in the new design. By our definition for x', this implies that $x' = x$ when m is even and $x' = x^C$ when m is odd. The argument for R_V is similar.

For R_D, we note that a reflection along the positive diagonal swaps the positions of x and y. For R_A, we find that $(y')^R$ becomes the new horizontal word and $(x')^R$ becomes the new vertical word. Coupled with our definitions for x' and y', the result quickly follows. □

Lemma 3. *The counterclockwise rotations r_{90}, r_{180}, and r_{270} have the following effects on the binary strings that generate a GHP:*

$$r_{90}(x,y) = \begin{cases} (y^R, x) & m \textit{ even} \\ (y^R, x^C) & m \textit{ odd} \end{cases}$$

$$r_{270}(x,y) = \begin{cases} (y, x^R) & n \textit{ even} \\ (y^C, x^R) & n \textit{ odd} \end{cases}$$

$$r_{180}(x,y) = \begin{cases} (x^R, y^R) & n, m \textit{ even} \\ (x^{RC}, y^R) & n \textit{ even, } m \textit{ odd} \\ (x^R, y^{RC}) & n \textit{ odd, } m \textit{ even} \\ (x^{RC}, y^{RC}) & n, m \textit{ odd} \end{cases}$$

Proof. For r_{90}, y^R becomes the new bottom word and x' becomes the new left word. For r_{180}, $(x')^R$ becomes the new bottom word and $(y')^R$ becomes the new left word. For r_{270}, y' becomes the new bottom word and x^R becomes the new left word. Combined with our definitions for x' and y', the results follow. □

To see if a GHP possesses particular symmetries merely from the defining binary strings, we can make use of our characterizations in Lemmas 2 and 3 in conjunction the following lemma.

Lemma 4. *Let $a \in \{0,1\}^n$. Then for all n, $a \neq a^C$. Additionally, if n is odd, then $a \neq a^{RC}$.*

Proof. It is immediately obvious that $a \neq a^C$, since this would require $0 = 1$. When n is odd, then $a_{(n+1)/2}$ is the entry in the center of a. If $a = a^{RC}$, then $a_{(n+1)/2} = 1 - a_{(n+1)/2}$, which is again impossible. □

A symmetry for a GHP will be an isometry that fixes the pair of binary strings. Given this view, we can combine Lemmas 2, 3, and 4 to arrive at the following theorem, which characterizes the symmetries of a GHP generated by two binary strings.

Theorem 5. *Consider a GHP generated by binary strings (x,y) where $|x| = n$ and $|y| = m$. Then:*

- *R_H (resp. R_V) is a symmetry if and only if $y = y^R$ (resp. $x = x^R$) and m (resp. n) is even*
- *R_D is a symmetry if and only if $n = m$ and $x = y$*
- *R_A is a symmetry if and only if $x = y^R$ when $n = m$ is even and $x = y^{RC}$ when $n = m$ is odd*
- *r_{90} and r_{270} are symmetries if and only if $n = m$ is even and $x = x^R = y = y^R$*
- *r_{180} is a symmetry if and only if one of the following is true: n and m are even, $x = x^R$, and $y = y^R$; n is odd, m is even, $x = x^R$, and $y = y^{RC}$; or n is even, m is odd, $x = x^{RC}$, and $y = y^R$*

An important consequence of Theorem 5 is a restriction on how rotations and reflections must relate.

Corollary 6. *Consider a point P in a GHP (with a potentially unbounded motif).*

1. *P is a center of 4-fold rotational symmetry if and only if P lies at the intersection of 4 lines of reflective symmetry.*
2. *P is a center of 2-fold rotational symmetry that is not on gridlines if and only if P lies at the intersection of at least 2 lines of reflective symmetry.*

Proof. In (1), either of the provided conditions implies that P is at the center of one of the squares created by our grid, since that is the only way these symmetries can exist. So for (1) and (2), even if we are working with an unbounded motif, we may consider an arbitrarily large square patch with P at the center and whose boundaries lie on gridlines. The symmetries of this arbitrarily large patch will be inherited by unbounded motif. The positioning of P implies the cut-out will have odd-dimensions and hence be generated by two binary strings of even length.

For (1), Theorem 5 gives us that the conditions for r_{90} and r_{270} to be symmetries imply R_H, R_V, R_D, R_A are all symmetries and vice-versa. For (2), the conditions for r_{180} where n, m are even are the same as those to obtain both R_H and R_V. □

An important implication of Corollary 6 is that a GHP can never exhibit the symmetries of C_4 (i.e. the group of rotations of the square with no reflective symmetry).

Wallpaper Groups of Periodic Generalized Hitomezashi Patterns

As we turn our attention to the 17 wallpaper groups, we can discard any group with 3- or 6-fold rotational symmetry, since these are incompatible with our grid of stitches. Of the remaining 12 groups, only 9 are attainable. In this section, we provide proofs for why pgg, p4, and p4g are not attainable symmetries in the framework of hitomezashi. In addition, we provide examples of the 9 attainable wallpaper groups.

A GHP with two distinct directions of translation symmetry, must be generated with two periodic binary words. For instance, we could take $x = \ldots 0111001110 \ldots = (01110)^\infty$ and $y = \ldots 10111011 \ldots = (1011)^\infty$, and assign each entry of these periodic words to an integer point on the x- or y-axis. We populate our stitches in all directions from that assignment. In this case, we say that x has period 5 and y has period 4. We call these *periodic generalized hitomezashi patterns* (PGHPs for short).

Theorem 7. *The wallpaper groups p4 and p4g cannot be realized in a PGHP.*

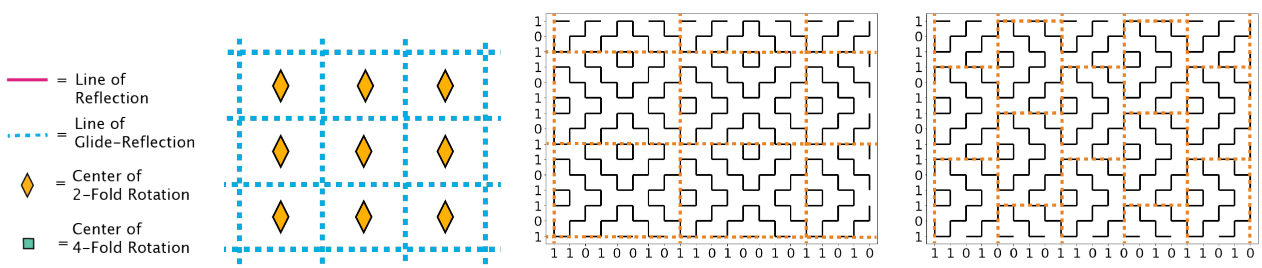
Proof. Both p4 and p4g have a center of 4-fold rotational symmetry without intersecting lines of reflection. By Corollary 6, we know that this is impossible to obtain in any GHP. □

All that remains is to show pgg is impossible. We provide a diagram of the symmetries of pgg in Figure 4b. To simplify our arguments, it is useful to break our periodic pattern into a tiling of finite, rectangular, GHPs.

Lemma 8. *A periodic generalized hitomezashi design can be viewed as an edge-to-edge tiling of identical, rectangular tiles with even-length dimensions. Further, there exists a smallest such tile so that every other such tiling will have rectangles with length and width greater than or equal to the smallest tile.*

Proof. The rectangular tile must have even dimensions since we are alternating stitches above and below the fabric. Additionally, since PGHPs are generated from periodic binary strings, we know there will be *some* rectangle that is large enough to contain the repeated pattern.

Note that the dimensions of the rectangular tiles must be a multiple of the minimal period length of our generating binary strings. It is straightforward to check that the smallest rectangle will have length/width equal to the period length when it is even and twice the period length when it is odd. □



(a) (b) (c) (d)

Figure 4: *Figure (a) gives the key for our symmetry markings, Figure (b) shows the symmetry diagram for pgg, Figure (c) shows the partition of a wallpaper design by Lemma 8. and Figure (d) shows a non-edge-to-edge tiling of the same design.*

An example of Lemma 8 can be seen in Figure 4c. Here the period of the horizontal word is 8 and the period of the vertical word is 3, so our rectangles have dimension 8×6. One thing to note is that the tiling described won't necessarily yield the smallest possible tiling of rectangles once we remove the edge-to-edge condition, which can be seen in Figure 4d.

We are able to shift the borders of our rectangular tiles in any direction—the pattern of stitches on the cells is not unique. However, it is useful to consider aligning the grid so that each cell can be thought of as a GHP generated by finite-length strings. In this case, suppose that our cell has dimensions $k \times \ell$. Then we can associate two binary words $|x| = k + 1$ and $|y| = \ell + 1$ with the added requirement that $x_1 = x_{k+1}$ and $y_1 = y_{\ell+1}$, since the boundaries of our cells overlap.

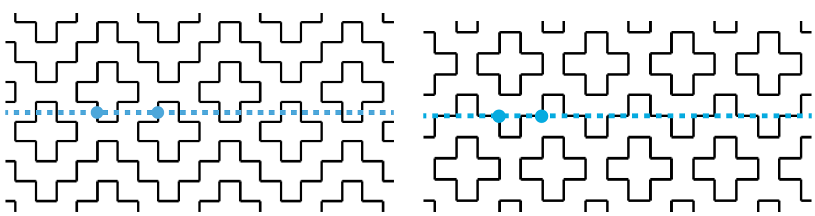
Since pgg is largely comprised of glide-reflective symmetries, we need to do some work to understand the behavior of glide-reflective symmetries in PGHPs.

Lemma 9. *A horizontal or vertical line of glide-reflective symmetry lies on (resp. off) a gridline if and only if the distance of glide is even (resp. odd).*

Proof. Naturally, any distance of glide on a horizontal or vertical line must be of integer length. In this proof, we will assume our line of glide-reflection is horizontal since the vertical case is symmetric.

If the line of glide-reflective symmetry lies off a gridline, then the vertical stitches exhibit reflective symmetry. So, just the translation of the glide-reflection is necessarily a symmetry for the vertical stitches. To create a non-trivial glide-reflection, performing just the glide must *not* be a symmetry for the horizontal stitches. By the alternating nature of the stitching, this can only happen if the distance of our glide-reflection is odd, not even.

Now suppose we have a horizontal line of glide-reflection with a glide of odd distance. If this line lies on a gridline, there is a line of horizontal stitches on the line of glide-reflection that gets translated some distance, but then fixed by the overlapping line of reflection. This means an odd-length glide cannot be a symmetry, since the stitches alternate over and under the fabric. Therefore the line of glide-reflection must lie off of a gridline. □



(a) (b)

Figure 5: *Examples of lines of glide-reflection where Figure (a) has odd-length glide and Figure (b) has even-length glide. The dots mark the smallest glide distance.*

Lemma 10. *If a PGHP has a line of glide-reflective symmetry with slope* ± 1*, then it also has lines of reflection with the same slope.*

Proof. We only consider the case when a PGHP has a line of glide-reflection with slope $+1$, since the case with slope -1 is symmetric.

The tiling with the smallest tiles as described in Lemma 8 must be an edge-to-edge tiling of $2k \times 2k$ squares for some integer k. This follows because the glide-reflection will exchange the length and width of the tiles, and these dimensions were unique. Consider one of the tiles, positioning it so that the line of glide-reflection is its diagonal.

We first consider the glide distance of the glide-reflective symmetry. A glide-reflection that moves the square to line up with another would imply that we are actually looking at a line of reflection. This means

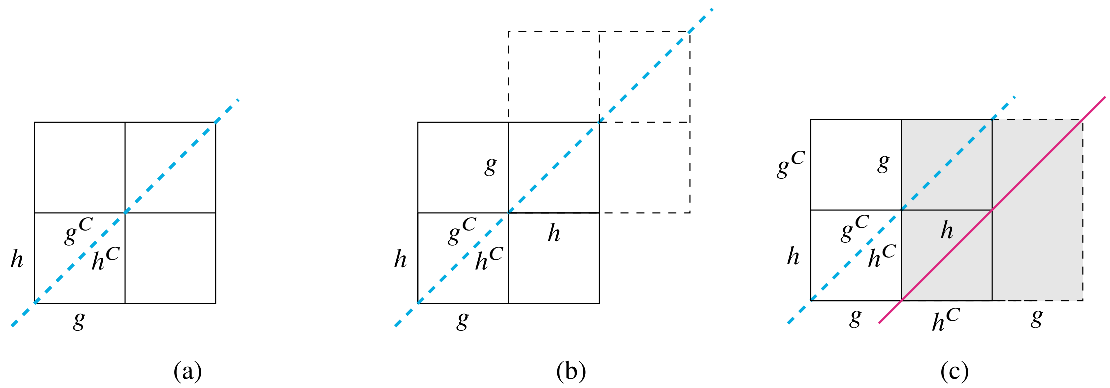


Figure 6: *Images of one of our $2k \times 2k$ lattice squares for the proof of Lemma 10. This displays the case where k is odd.*

the glide distance must be shorter than the diagonal of the square. Since performing the glide-reflection two times gives a translation, it will naturally follow that the glide distance is half the length of the diagonal.

Our argument is illustrated in Figure 6. We break our tile into four $k \times k$ squares that share borders; our glide-reflection (or its inverse) will map the lower left square onto the upper right square. Each quarter can be generated with two binary strings of length $k+1$. Let the bottom-left square be generated by $g, h \in \{0, 1\}^{k+1}$.

By the alternating nature of our stitches, we can label the edges of the other squares that lie parallel. The parity of the distance between the parallel lines matters: when k is even, the label is the same; for k odd, we must use the complement of the label. Figure 6a shows this labeling for the case when k is odd. When k is even, every g^C and h^C is replaced with g and h, respectively, which is somewhat simpler. For this reason, we only work with the odd case and leave the even case to the reader.

When we apply the glide-reflection, we can find more labels for our square as shown in Figure 6b. From there we can determine that our chosen tile is generated by gh^C and hg^C, with the caveat that the last entry of g (resp. h^C) must be the first entry of h^C (resp. g) since these entries overlap. With this information, we see that if we shift the borders of our tile to the right by a distance of k, as in Figure 6c, then the new square that tiles the plane is generated by the strings $h^C g$ and $h^C g$—that is, our tile exhibits R_D reflective symmetry. □

Theorem 11. *The wallpaper group pgg cannot be realized in a periodic generalized hitomezashi design.*

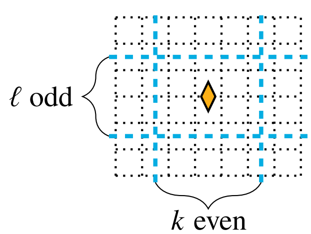


Figure 7: *An illustration of how lines of glide-reflection would relate to a 2-fold center of symmetry in pgg.*

Proof. We cannot realize a pattern with symmetry type pgg where the lines of glide-reflection have slope ± 1 since this will also imply there are lines of reflection by Lemma 10. So we only consider glide-reflections that are parallel to the x- and y-axes.

Since pgg has centers of 2-fold rotation with no lines of reflection, the center cannot be off the gridlines by Lemma 6. Assume the center of rotation lies on a horizontal gridline; the other case is symmetric.

The lines of glide-reflection partition the plane into rectangles where each 2-fold center of rotation is in the center of a rectangle. Note that these rectangles are actually examples of the tiles described in Lemma 8, so must have even dimension. Suppose the width of each of these rectangles is k, where k is even. This means that the vertical lines of glide-reflection must lie off of the gridlines, and by Lemma 9, the glide distance along the vertical lines is odd. This is a contradiction, since the rectangles must have an even-length height. Therefore, pgg is not a possible symmetry type. □

We are left with nine possible wallpaper groups, all of which are illustrated in Figure 8. We mark the symmetries (leaving out translation symmetry) on a portion of each design using the symbols in Figure 4a.

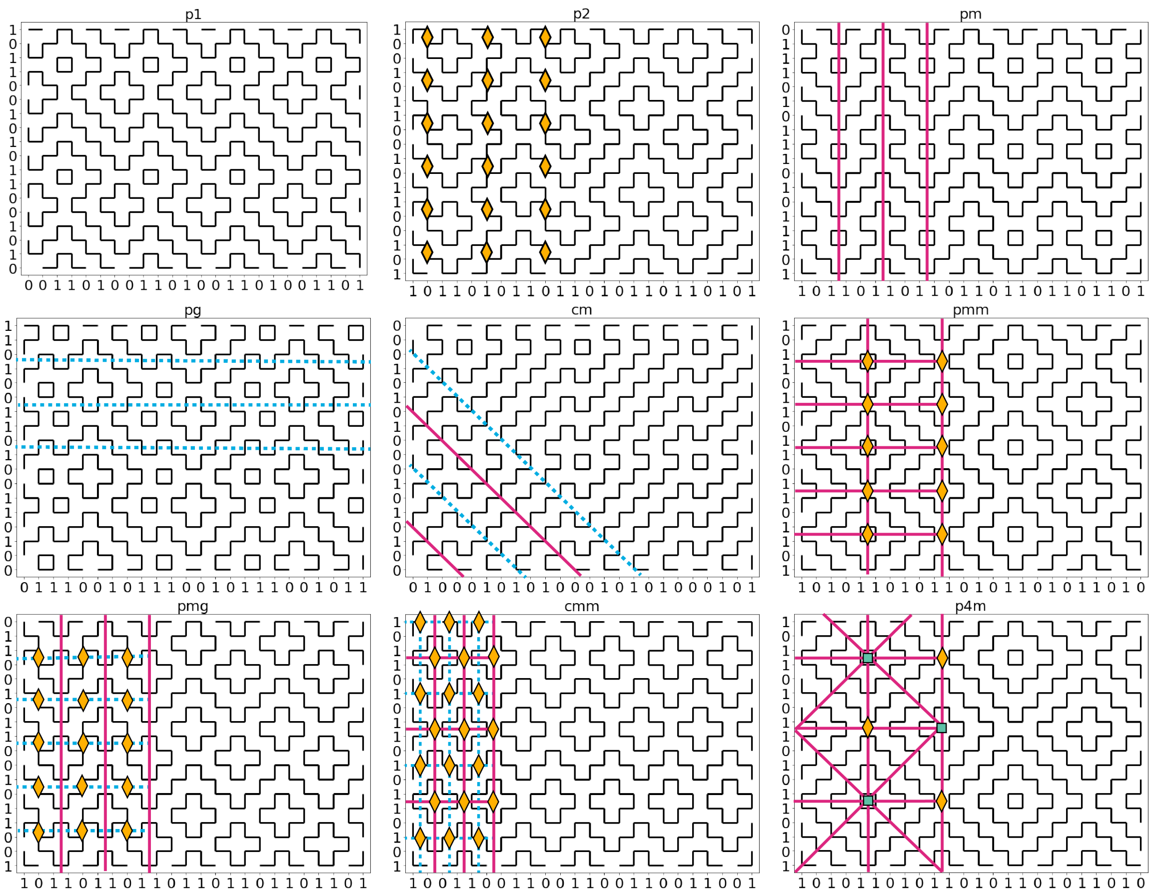


Figure 8: *Examples of the 9 attainable wallpaper paper symmetries in PGHPs.*

References

[1] S. Briscoe. *The Ultimate Sashiko Sourcebook: Patterns, Projects and Inspirations*. David & Charles, 2020.

[2] C. Defant and N. Kravitz, "Loops and regions in Hitomezashi patterns." *Discrete Mathematics*, vol. 347, no. 1, 2024, 113693.

[3] S. Goldstine, "A Survey of Symmetry Samplers." *Bridges Conference Proceedings*, Waterloo, ON, July 27-31, 2017, pp. 103-110.

[4] C. Hayes and K. A. Seaton, "A two-dimensional introduction to sashiko." *Bridges Conference Proceedings*, Virtual Conference, 2020, pp. 517–524.