Rotated Grids for Origami Tessellation Pattern Alignment

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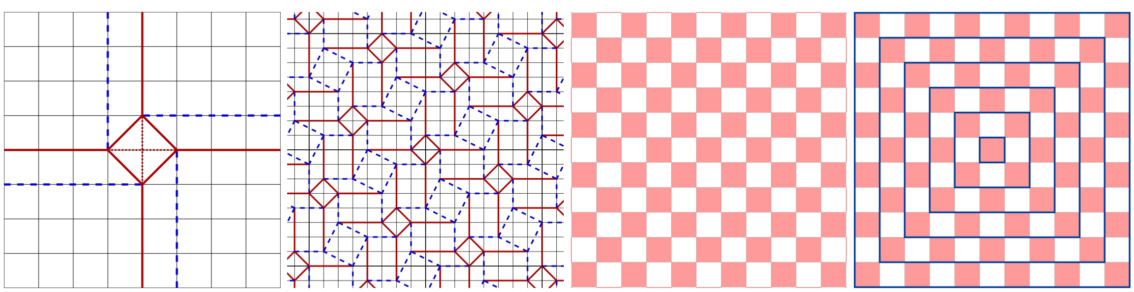
Abstract

When folding origami tessellations by hand a background grid is typically folded parallel to the edges of the paper, whether on a square or on a hexagon. However, not all pattern repeats align with the grid and so a rotation of the grid relative to the edge of the paper is required to align pattern repeats or other pattern features with the edge of the paper. The calculations needed to perform this alignment are remarkably elegant, with no angles of rotation needed. This paper will prove the equations and demonstrate the methods for rotated grids on squares and hexagons to achieve pattern alignment with the edge of the paper.

Introduction

Origami tessellations are finite expressions of an infinite tessellation pattern, folded from a single sheet of paper [2]. They obey geometric constraints imposed by the paper and the dominant structure used in flat-foldable origami tessellations is a **twist**, a central polygon of folds surrounded by pleats composed of a parallel mountain and valley fold as in Figure 1a [1]. These pleats then connect to the neighboring twist, which will have the opposite sense of rotation as the first twist. In the particular case of Figure 1b, two square twists are alternating to make a **crease pattern**, a map of what lines will be folded in the origami tessellation.



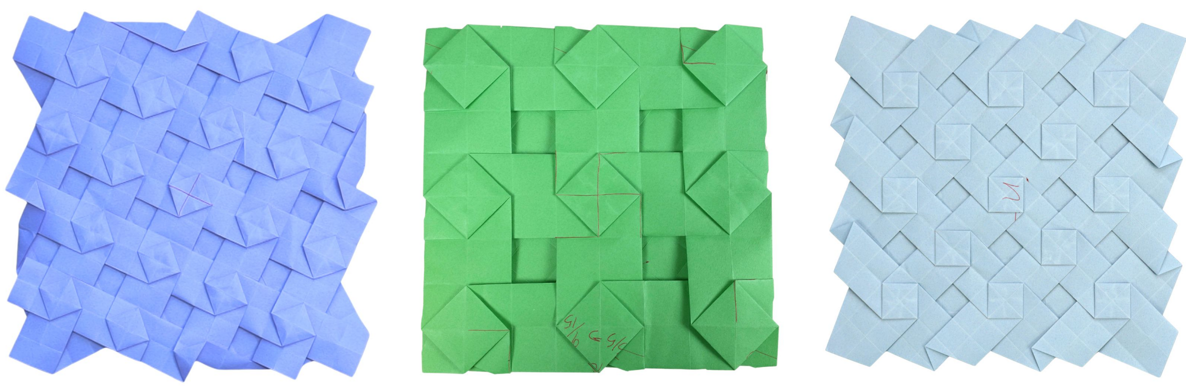
(a) (b) (c) (d)

Figure 1: *Tessellation definitions (a) Twist, closed square, (b) Crease pattern with alternating square twists on opposite sides of the paper, (c) Square tiling, (d) Square tiling with a set of tiling breaks.*

Each twist has pleats that go in certain directions (e.g. four pleats each spaced 90 degrees apart) and can be abstractly described by a **tile** with edges perpendicular to the folds of the pleat. **Tilings** are arrangements of these tiles and have properties that apply to all tessellation patterns that match that connectivity of twist shapes, regardless of whether the tiling maps directly onto the crease pattern. For example, there is no mapping of the tiling in Figure 1c to the crease pattern in Figure 1b without overlaps or gaps (assuming tile edges are perpendicular to pleats), yet the connectivity of each square to four square neighbors is the same. One of those properties is whether there exists a set of twists in the tiling with a convex outline, which mandates that none of the pleats leaving that set of twists will intersect with another since all pleats cross tile boundaries perpendicularly. I call these convex outlines **tiling breaks** and a concentric set of them are shown in Figure 1d. This definition could be expanded to any line that passes straight through a tiling entirely on tile boundaries.

When origami tessellations are folded by hand, a background grid is used to pre-fold the pleats between twists and establish the reference points for the central polygons of the twists. This allows the folder to add one twist to the paper at a time, setting up the pleats and then squashing the center to fold the central polygon. The entire tessellation can be folded by sequentially adding twists to the paper until either all the paper is used or the folder arrives at a desired stopping point. Common methods for adding the twists to the paper include working in concentric rings around the center and working in stripes, whether from the center or from a corner.

When folding all the way to the edge of the paper it is common for the central polygon of twists to be interrupted by the edge, leaving a jagged, unpleasant boundary around the neatly ordered center as in Figure 2a. A much more visually appealing alternative is to stop folding at a tiling break before reaching the edge, creating a pleated border all the way around the tessellation, especially when the tessellation repeats are also aligned with the edge of the paper and the pleated border is the same width all the way across each edge as in Figure 2c. However, pattern repeats may be aligned with the edge of the paper without the tiling break also being aligned as in Figure 2b and this is the best option available for patterns in tilings without tiling breaks and patterns whose repeats do not line up with any tiling breaks.



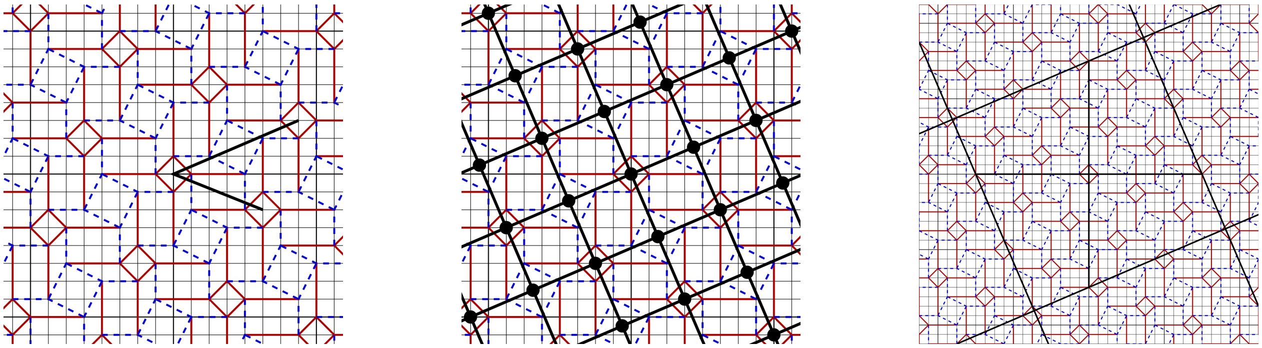
(a) (b) (c)

Figure 2: *Folding alignment options (a) Grid aligned with the edge of the paper, (b) Closest pattern repeats aligned with the edge of the paper, (c) Tiling breaks aligned with the edge of the paper.*

Alignment

Now that we've established that alignment of pattern repeats and/or tiling breaks with the edge of the paper is artistically preferable for origami tessellations, it's time to examine how to achieve this outcome. The tiling of the pattern determines whether a pleated border is possible through the presence or absence of tiling breaks and a particular pattern may or may not have closely spaced repeats aligned with any tiling breaks that are present. Closely spaced repeats are specified here to keep the project reasonably small.

Both tiling breaks and lines between pairs of repeats have a fixed geometric relationship with the background grid of the origami tessellation crease pattern as seen in Figure 3a and we can overlay the dual of the tiling onto the crease pattern as in Figure 3b to see which of those alignment options matches the tiling breaks. Then we can count out from our desired center point past our desired number of pattern repeats and draw a boundary matching the slope of our target alignment that meets the grid lines coming out from the center at grid intersections as in Figure 3c. This is the subset of the infinite tessellation that was chosen to be folded for Figure 2c, and the process described works exactly the same way for square grids on square paper as it does for equilateral triangle grids on hexagonal paper.

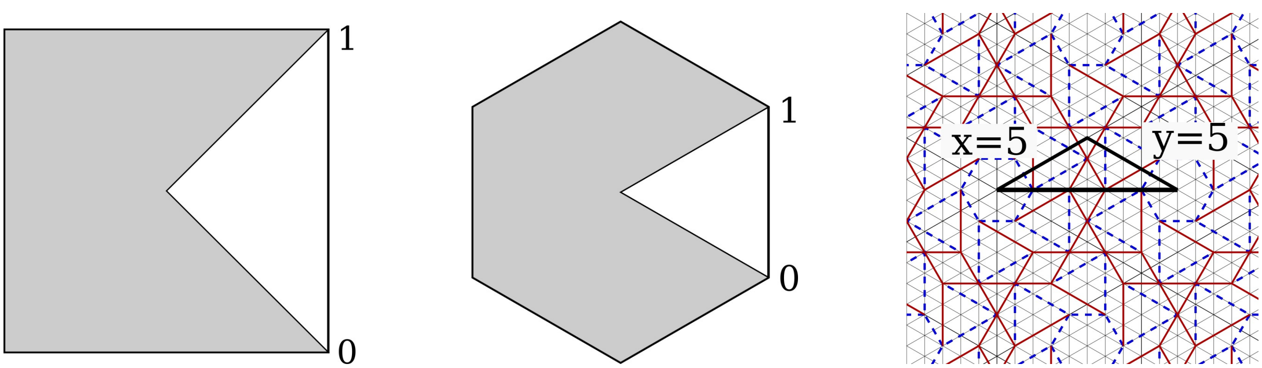


(a) (b) (c)

Figure 3: *Alignment on a crease pattern (a) Two densest options for alignment of repeats, (b) Dual of tiling on crease pattern, (c) Outline of desired tiling-aligned section of crease pattern.*

Notice that the grid lines running through the chosen pattern center partition the edges of the outlined square into two parts in Figure 3c. The relative proportions of these two parts of the square boundary are invariant for a given alignment regardless of how many grid divisions are counted from the center to construct the boundary - consider that for any given number of divisions, you can always divide each division in half to double the number of divisions without changing existing proportions. These relative proportions can be calculated manually by drawing slopes on grid paper, but an exact fraction is desired and the calculation becomes burdensome for most slopes. The next section derives and gives the formulas for how to calculate where the division point between these partitions lies relative to the length of the boundary edge given only the countable components of the slope of the alignment on the background grid for square and hexagon cases.

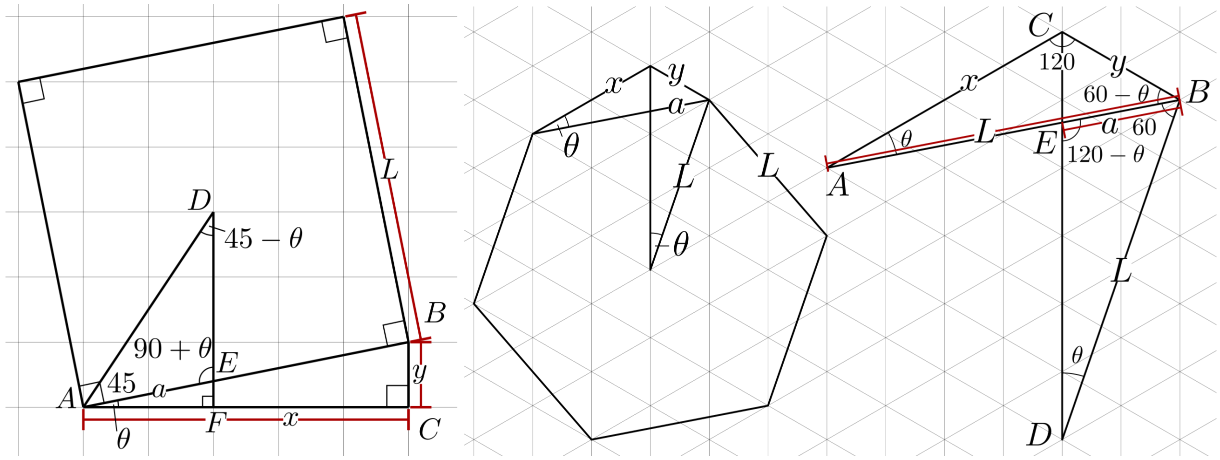
Grid Intersection Centered



(a) (b) (c)

Figure 4: *Bounds and counting on hexagons (a) Square grid function domain and range, (b) Triangle grid on a hexagon domain and range, (c) Slope components on a triangle grid.*

Given a slope on a background grid, what is the fraction of the right-side edge where a grid line that passes through the center of the paper will touch the edge given that the edge matches the given slope? Let's first consider the case of square grids on square paper as in Figure 5a. The input is the x and y components of a slope, where $-x \le y \le x$ and the output should be a number between 0 and 1 as shown in Figure 4a, expressed as a fraction. We see sample slopes in Figure 3a, where the upper slope has $x = 7, y = 3$ and the lower slope has $x = 5, y = -2$.



(a) (b)

Figure 5: *Setups for alignment proofs (a) Square grids on a square, (b) Triangle grids on a hexagon.*

Now consider the square that's been rotated relative to a background square grid to match the target slope in Figure 5a. Point D is the center of the square, $\overline{DF}$ is perpendicular to $\overline{AC}$, $\overline{AD}$ is the angle bisector of the corner of the square at A, the length of the edge of the square is L, and the length of $\overline{AE}$ is a. Our goal is to calculate a/L. Labelling $\angle BAC = \theta$, we calculate that $\angle ABC = 90 - \theta$, $\triangle ABC \sim \triangle AEF$, $\angle AED = 90 + \theta$, and $\angle ADE = 45 - \theta$. From $\triangle ABC$ we know that $\sin(\theta) = y/L$ and $\cos(\theta) = x/L$. By the Law of Sines on $\triangle ADE$, $\sin(45 - \theta)/a = \sin(90 + \theta)/(L/\sqrt{2})$, and so our target fraction becomes:

$$\frac{a}{L} = \frac{\sin(45 - \theta)}{\sqrt{2}\sin(90 + \theta)} = \frac{\sin(45)\cos(\theta) - \cos(45)\sin(\theta)}{\sqrt{2}(\sin(90)\cos(\theta) + \cos(90)\sin(\theta))} = \frac{x/L - y/L}{2x/L} = \frac{x - y}{2x} = \frac{1}{2}(1 - \frac{y}{x}) \quad (1)$$

We can then verify that the range $y : [-x, x]$ maps to $a/L : [0, 1]$ for positive values of x, keeping in mind that outputs 0 and 1 are identical for our purposes and that y and a/L are negatively correlated.

Next, we'll turn to the case of an equilateral triangle grid on a hexagon. Here we define x and y as the counts on grid lines above a slope in the white range of Figure 4b and with a 120 degree angle between grid lines as seen in Figure 4c. The boundary of the white range from the center to the corner marked 0 corresponds to $x = 0$ and the boundary to the corner marked 1 corresponds to $y = 0$. Only positive values are permitted for x and y in this situation.

Consider the hexagon that's been rotated relative to a background equilateral triangle grid to match the target slope in Figure 5b. The relevant portion of the diagram is shown larger on the right. Point D is in the center of the hexagon, both $\overline{AB}$ and $\overline{DB}$ have length L, $\overline{BE}$ has length a, and $\angle BAC$ is defined as θ. We know from the grid and the initial hexagon that $\angle ACB = 120\,\mathrm{deg}$, $\angle ABD = 60\,\mathrm{deg}$, and that $\overline{CD}$ is an angle bisector of $\angle ACB$. From there we can derive that $\angle ABC = 60 - \theta$, $\angle BEC = 60 + \theta$, $\angle BED = 120 - \theta$, and $\angle BDE = \theta$. Our goal is to calculate a/L for this new context. By the Law of Sines on $\triangle ABC$ we get that $\sin(\theta)/y = \sin(120)/L$ and therefore $\sin(\theta) = \sqrt{3}y/2L$. Also by the Law of Sines on the same triangle:

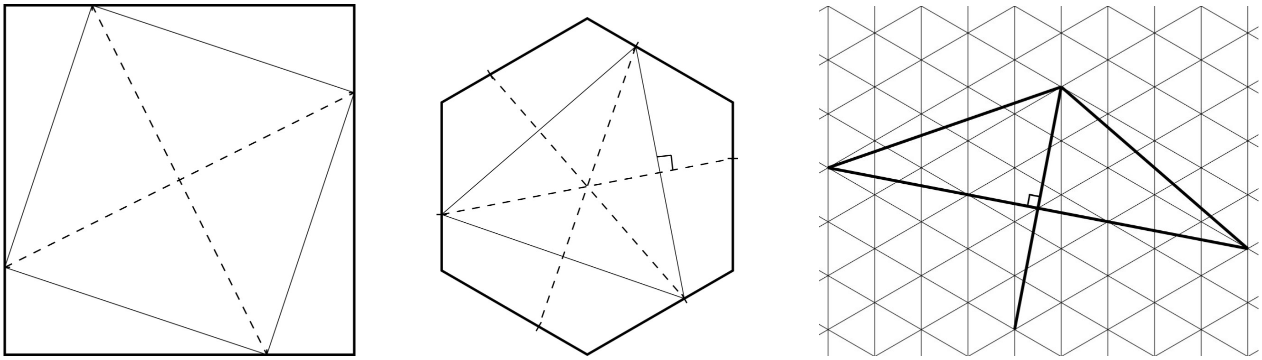
$$\frac{\sin(\theta)}{y} = \frac{\sin(60 - \theta)}{x} = \frac{(\sqrt{3}/2)\cos(\theta) - (1/2)\sin(\theta)}{x} = \frac{\sqrt{3}\cos(\theta) - \sin(\theta)}{2x}, \quad \cos(\theta) = \frac{2x + y}{2L}$$

By the Law of Sines for $\triangle BED$, $\sin(\theta)/a = \sin(120 - \theta)/L$ and so:

$$\begin{aligned}\frac{a}{L} &= \frac{\sin(\theta)}{\sin(120 - \theta)} = \frac{\sin(\theta)}{(\sqrt{3}/2)\cos(\theta) + (1/2)\sin(\theta)} = \frac{\sqrt{3}y/2L}{(\sqrt{3}/2)(2x + y)/2L + (1/2)\sqrt{3}y/2L} \\ &= \frac{y}{(2x + y)/2 + y/2} = \frac{y}{x + y}\end{aligned} \quad (2)$$

We can then verify that $y = 0$ yields 0 and that $x = 0$ yields 1, and that when $x = y$ then $a/L = 1/2$.

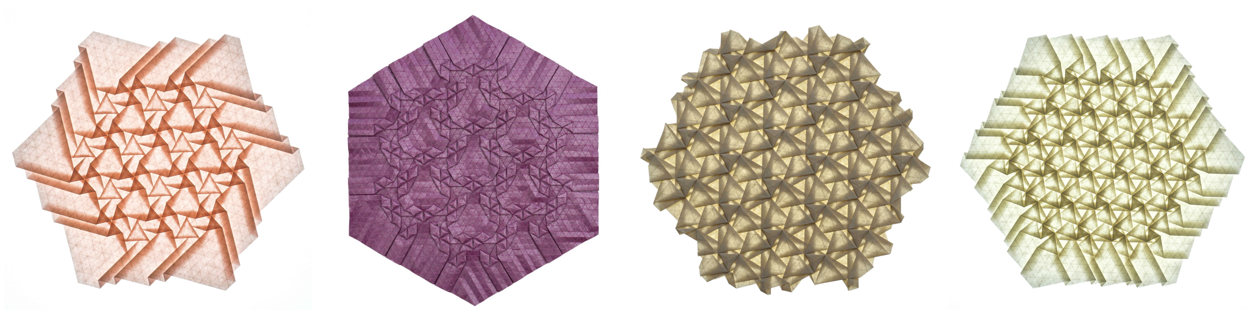
Grid Shape Centered



(a) (b) (c)

Figure 6: *References for grid shape centers (a) Square grids on square paper, (b) Triangle grids on hexagonal paper, (c) Finding the perpendicular on triangle grids.*

The last section focused on grids whose central axes intersect at the center of the paper, but that's not the only option for rotational symmetry in the center. Grid squares (or grid triangles) are also valid places to center a tessellation and there are some origami tessellations that don't have any positions of rotational symmetry on a grid intersection. Thankfully, there's a shortcut to find the a/L references needed for these cases - find the diagonal of your target grid alignment and calculate the grid-intersection-centered references for that line. The diagonal can be found by making an isosceles triangle with two copies of the target alignment line laid out as if they were adjacent sides of the paper shape and drawing the diagonal between the two free corners. We see this process in Figure 6c, where our target alignment has $x = 4$, $y = 1$ while the diagonal has $x = 1$, $y = 2$. A third orientation of the target alignment is included to demonstrate that it is perpendicular to the diagonal, as shown in Figure 6b.

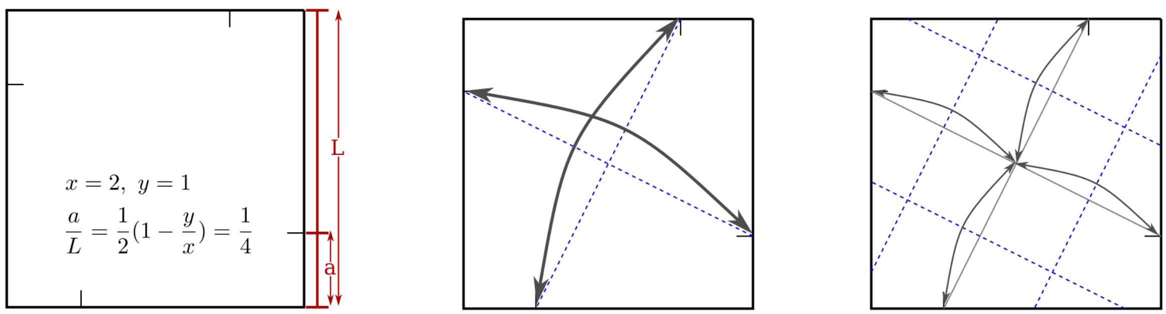


(a) (b) (c) (d)

Figure 7: *Triangle-centered origami tessellations (a) Edge-aligned grid, (b) Edge-perpendicular grid, (c) Wrong number of grid divisions, (d) Aligned.*

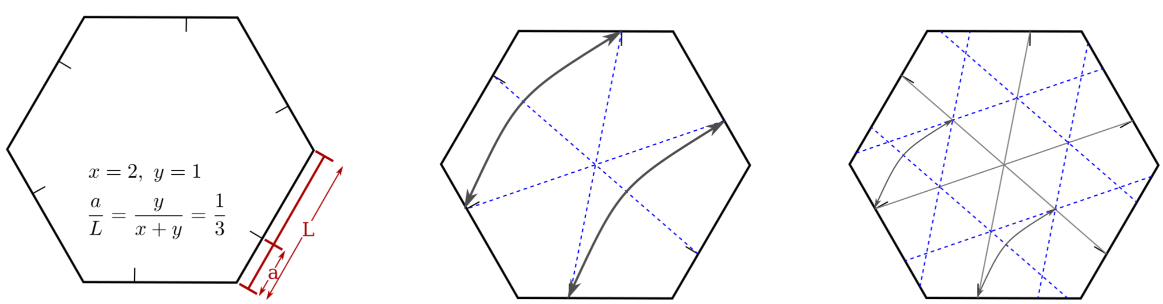
As with the grid-intersection-centered rotated grids, there are two options that don't require extra calculations: the edge-parallel and edge-perpendicular (diagonal in the square grid case) options. Examples of these are shown in Figure 7 along with a failed and a aligned version of the same tessellation. There are a few extra considerations for your target number of divisions when using shape-centered grids. The first rule is that you must not fold new grid divisions through the center of your grid shape. On square grids that means no even divisions and on triangle grids that means no divisions by three. Additionally, the central grid triangle will flip orientation with each round of grid refinement and so you'll need to perform an even number of grid refinement steps. This condition does not apply to the edge-parallel and edge-perpendicular cases since those setups are mirror symmetric.

Application



(a) (b) (c)

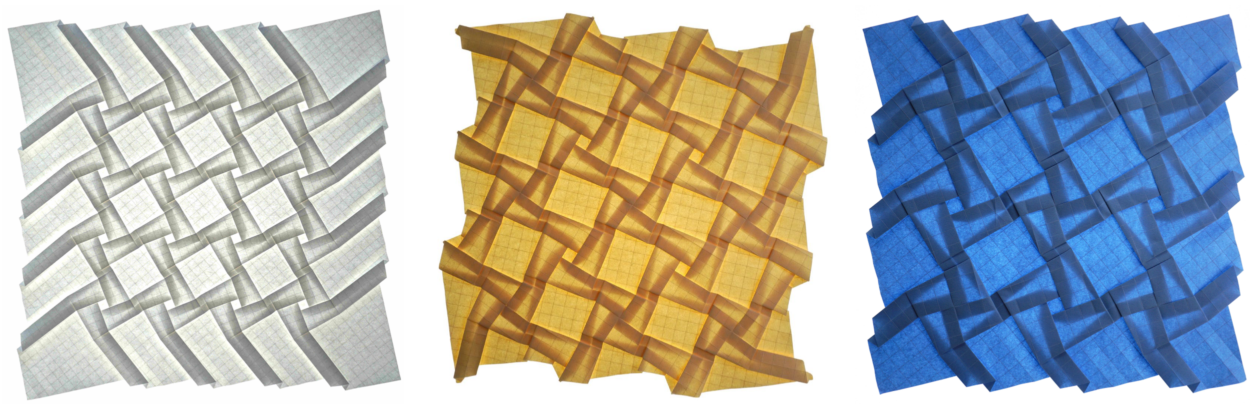
Figure 8: *Folding a rotated square grid (a) Calculate a/L and pinch references, (b) Fold central grid lines, (c) Perform further divisions.*



(a) (b) (c)

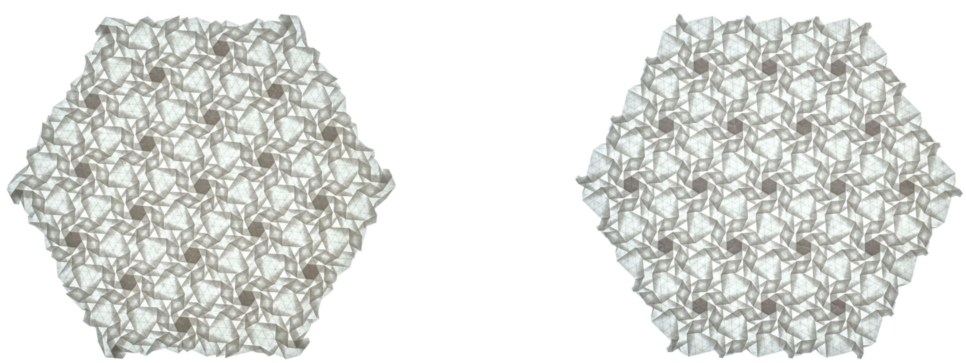
Figure 9: *Folding a rotated triangle grid on a hexagon (a) Calculate a/L and pinch references, (b) Fold central grid lines, (c) Perform further divisions.*

The steps to fold a rotated grid are to find a desired alignment in the origami tessellation crease pattern, decide whether your center will be on a grid intersection or a grid shape, calculate your a/L reference points, mark those points on your paper, fold the main axes of the grid, and refine the grid until you reach your target number of divisions. Finding your desired alignment is most subject to personal taste and may require trial and error to get a sense of where you feel the alignment should be, as in Figure 10. When folding a rotated grid, there are two key pieces of information to keep in mind. First, as soon as you mark your a/L references your paper becomes chiral (assuming they aren't at 0 or 1/2) and it is imperative to keep track of which side is the front so that you don't end up wondering where you went wrong as in Figure 11a. Second, recall that we defined those reference points as grid intersections in the Alignment section and so those points are what need to come to the center line to fold more divisions. Additional concerns dictate your number of grid divisions when using shape-centered rotated grids, as was described in the previous section.



(a) (b) (c)

Figure 10: *Alignment on square tiling tessellations (a) Compound Squares Rotated, densest repeats and tiling aligned, (b) Mixed Squares Rotated, densest repeats aligned, (c) Mixed Squares Rotated, tiling aligned.*



(a) (b)

Figure 11: *Common Mistake Number One (a) Correctly gridded, folding started on wrong side, (b) Correctly gridded, folding started on right side.*

Conclusions

When folding origami tessellations by hand, rotating the grid relative to the edges of the paper can improve the aesthetics of your finished pieces. My personal preference is to align to tiling breaks when repeats are aligned with tiling breaks and typically to align to repeats when there are few repeats or when they are not aligned with tiling breaks, although I will return to tiling break alignment for very large projects. Figure 12 shows a variety of possible symmetry options on a single tiling and the alignments I chose to use in each case – see [3] for an explanation of how to find these different symmetry options. In summary, the accessible calculations for rotated grid reference points that were proven in this paper enable rapid origami tessellation project planning with professional-looking borders that elevate these mathematical curiosities to the realm of fine art.

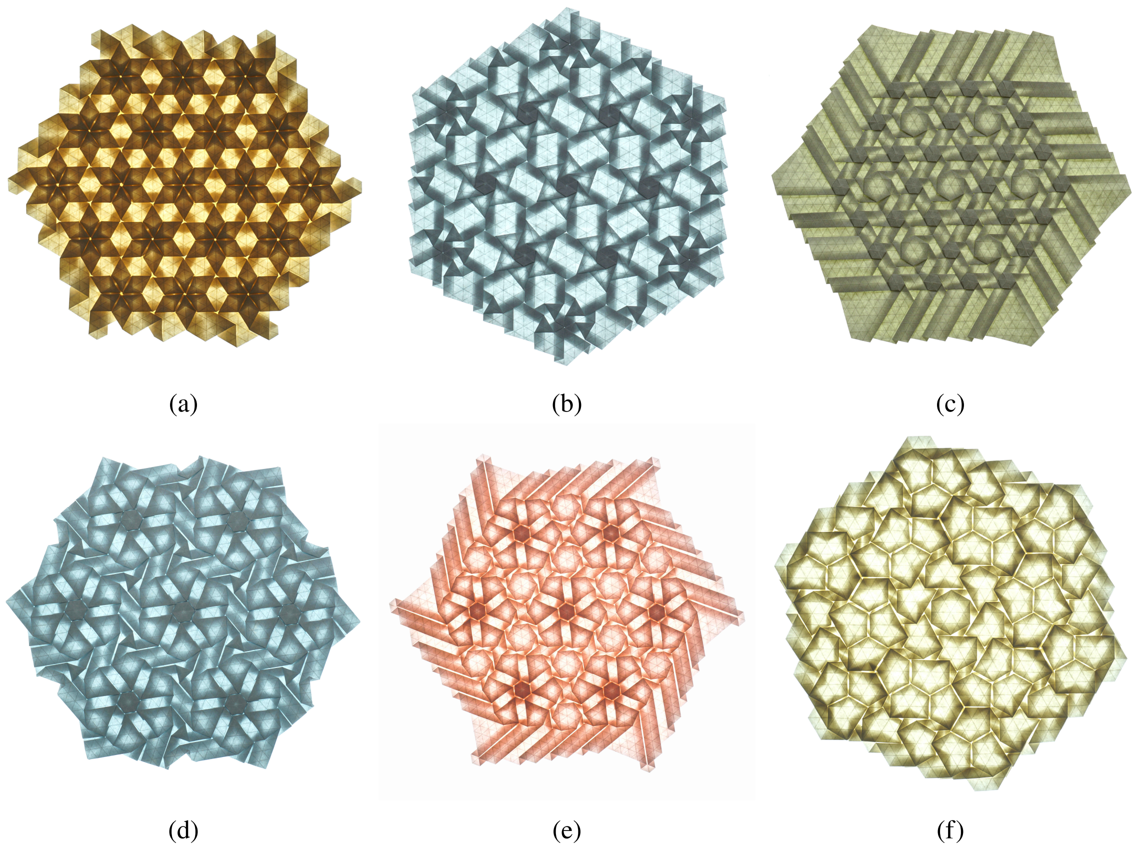


Figure 12: *Alignment and Symmetry Options (a) Alternating, tiling and repeat aligned, (b) First symmetry extension, tiling aligned with modified border, (c) Second symmetry extension, tiling and repeat aligned, (d) Third symmetry extension, repeat aligned, (e) Fourth symmetry extension, tiling and repeat nearly aligned on standard grid, (f) Far symmetry, tiling aligned.*

Acknowledgements

Thanks to Elseh, who introduced me to the concept of calculating rotated grids by drawing on grid paper in 2020.

References

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