Periodic Strips from Aperiodic Tiles

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Abstract

A periodic strip is a finite-width strip of tiles that repeats in one direction with frieze symmetry. They have many potential applications in art and design, particularly because of the ability to construct a finite portion of a periodic strip and wrap it seamlessly around a cylinder. I show that under very mild conditions, shapes that tile the plane also admit periodic strips of any desired width. This fact is true even for aperiodic tile sets. I explain why periodic strips exist, give simple methods for constructing them, and show examples for a variety of well known tilings.

Introduction

Although we think of a tiling of the Euclidean plane as an arrangement of shapes that extends to infinity in every direction, a drawing of a tiling is necessarily limited to a finite excerpt. The choice of excerpt naturally depends on the intended application, and might be influenced by both aesthetic considerations and pratical limitations in manufacturing techniques.

In this work I consider the construction of *strips* of tiles, arrangements of tiles that extend to infinity along a line in one direction, but are confined to a finite width in the perpendicular direction. A finite-width strip provides the raw material from which decorative friezes of tiles of any length might be extracted. To define strips precisely, recall first that a *slab* is a region of the Euclidean plane bounded by two parallel lines. A *strip* of tiles is an infinite collection of tiles (topological disks) such that:

- The tiles have disjoint interiors (they do not overlap);
- The union of the tiles contains some slab $\mathcal{S}_i$, and is contained in another slab $\mathcal{S}_o$; and
- The union of the tiles is simply connected, and the interior of the union is connected.

The third condition ensures that the strip's boundary is made up of exactly two curves. Thus tiles do not enclose any internal holes or form subsets connected to the rest of the strip by a single point.

In a typical design context, we will want to construct a strip that covers a slab of some desired width w, and will not object if the strip extends a bit farther on either side of that slab. Therefore, rather than measuring the "exact" width of a strip, I will say that a *strip of width w* is a strip containing a slab $\mathcal{S}_i$ of width *at least w*.

A strip is *periodic* if it has a translational symmetry. A periodic strip's translation symmetries must necessarily all run parallel to its slabs, meaning that the complete set of its symmetries will form one of the seven frieze groups [3, Section 1.4]. The *period* of a periodic strip is the length of its shortest translation symmetry vector.

In this paper I will demonstrate and discuss a fact that might seem counterintuitive at first: most sets of shapes that tile the plane also admit periodic strips of all widths. Here I use "most" informally to refer to tilings commonly studied mathematically or used in art and design; later I discuss the (mild) conditions under which a set of shapes will behave this way. If a set of shapes admits a periodic tiling, this fact is immediate: a periodic tiling already contains periodic strips of all widths. What is more interesting is that typical aperiodic tile sets also exhibit this behaviour, and furthermore that the existence of periodic strips does not contradict the aperiodicity of the tiles.

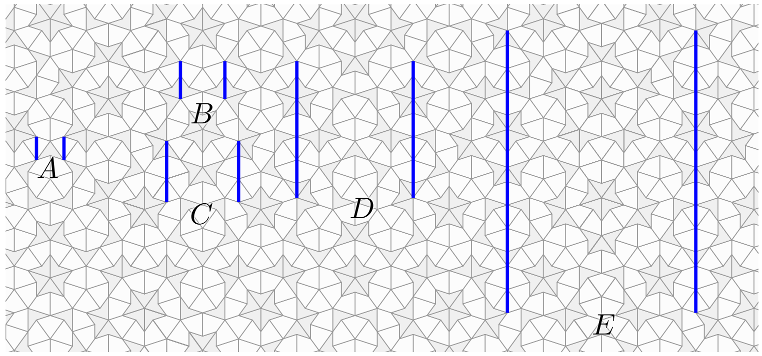

Figure 1: *A patch from a kite and dart tiling. Blue vertical lines indicate areas of local reflection symmetry. The lines come in labelled pairs that can be used to construct periodic strips.*

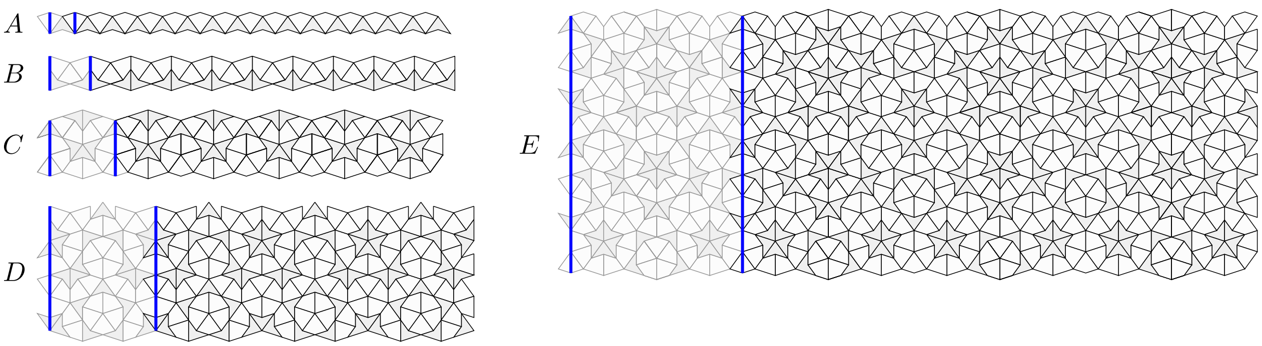

Figure 2: *Periodic strips based on the labelled pairs of lines in Figure 1.*

Periodic strips have many natural decorative applications. A periodic strip constructed from an aperiodic tile set preserves the visual character of that set's tilings and obeys any local matching rules, but avoids the complexity of working with a general strip. Furthermore, from any periodic strip we may extract a finite number of periods of the pattern and roll them up into a cylinder. That cylinder could serve as the design for an object like a lampshade or drinking glass. It could also be used in manufacturing, to apply arbitarily long friezes to materials. When Kimberly-Clark famously used Penrose rhombs as a quilting pattern for toilet paper [5], they may very well have adopted a similar approach. Other natural applications would be in printing fabric or wallpaper, or even in creating patterned cookies [4].

Kites and Darts

Before discussing more general constructions for periodic strips, I give a few explicit examples based on Penrose's kites and darts, one of the best known aperiodic tile sets [3, Section 10.3]. In a tiling by kites and darts, it is particularly easy to identify the building blocks of periodic strips by eye, giving us a quick way to see these ideas in action.

Figure 1 shows an excerpt from an infinite tiling by kites and darts. Any such tiling contains line segments that act as "lines of local reflection symmetry": ignoring any matching rules, the segment either passes through the line of reflection symmetry of a tile, or passes between a pair of tiles that are congruent through a reflection across the line. Figure 1 shows some vertical local reflection lines in blue. Each one extends as far as possible through areas of local reflection. The lines come in five labelled pairs, each pair consisting of two lines of equal length, offset horizontally, and encountering congruent sets of tiles along their lengths. It follows that each pair of lines defines one period of a periodic strip. Figure 2 shows periodic strips based on the five labelled pairs. Note that although the tiles are not drawn with the matching rules that are required to enforce non-periodicity, the tiles in these periodic strips obey those rules—no sneaky tricks are needed to create strips that repeat in the manner shown.

Any tiling by kites and darts must contain pairs of parallel line segments like these, with lengths that grow without bound. (The existence of these pairs follows from the substitution rules that define the tiling, but a full proof is beyond the scope of this paper.) Thus, for any desired width, a sufficiently large patch of kites and darts will contain a pair of lines suitable for constructing a periodic strip of that width.

The Existence and Construction of Periodic Strips

Kites and darts are not exceptional among tile sets in terms of their ability to admit periodic strips. In this section I argue—without offering a formal proof—that if a set of shapes admits any tilings at all, then we may generally assume that the set also admits periodic strips of all widths.

A tiling of the plane is called *repetitive* if, for any finite patch $\mathcal{P}$ of tiles, there exists a number $r > 0$ such that every disk of radius r in the plane contains a congruent copy of $\mathcal{P}$ [2]. In other words, no matter what finite patch of tiles you identify in the tiling, you are guaranteed to find another copy of that patch not too far away. Not every tiling is repetitive. But remarkably, Radin and Wolff showed that a set of shapes that tile the plane must admit at least one repetitive tiling [7].

For the construction of periodic strips we require a condition that is only slightly stronger. I will call a tiling *translationally repetitive* if it satisfies the definition of repetitivity above with “congruent” replaced by “translated”. That is, one may always find a translated copy of a patch within some bounded distance from that patch.

Any translationally repetitive tiling may be used to construct arbitrarily wide periodic strips. For any width w, place a disk of diameter w anywhere in the tiling. Now construct the smallest patch $\mathcal{P}$ of tiles containing that disk, and find a nearby translated copy $\mathcal{P}'$ of the patch. Finally, construct a new patch Q containing $\mathcal{P}$, $\mathcal{P}'$, and all the tiles “between” them, namely the tiles that intersect an oriented rectangle of width w connecting the two disks. We may then build a periodic strip by placing repeated copies of Q, aligning each one’s copy of $\mathcal{P}$ with its neighbour’s copy of $\mathcal{P}'$ and eliminating duplicate tiles. It is possible for this construction to fail if Q includes tiles that reach far around the boundaries of $\mathcal{P}$ or $\mathcal{P}'$. This issue can be avoided by constructing $\mathcal{P}$ and $\mathcal{P}'$ based on disks of diameter $w + d$, where d is chosen so that every tile shape is contained in a disk of diameter d.

In practice the construction need not be so generic. A more natural approach to be carried out by hand is to generate a large patch of tiles and then identify a sub-patch $\mathcal{P}$ and its nearby translation $\mathcal{P}'$ by eye. As before, we now build a repeatable patch Q from $\mathcal{P}$, $\mathcal{P}'$, and the tiles between them. The benefit of this approach is that we can choose patches to accomplish a design goal, like minimizing the period of the strip. The narrow patches of tiles surrounding lines of local reflection symmetry in Figure 1 can be seen as an example of this approach.

When a set of shapes is equipped with substitution rules [2], we can often take advantage of those rules to construct periodic strips even more easily, in a “bottom-up” manner. Here we do not require the full power of translational repetitivity a priori—it suffices for a sufficient number of substitutions to produce a patch containing two tiles with the same orientation. Begin with any tile shape as a seed, and apply the substitution rules until two such tiles T and T' appear. Draw a line segment parallel to the translation from T to T', with endpoints in their interiors. Construct a patch Q consisting of all tiles that intersect this line segment. This patch acts like Q above: we can repeat copies of it, overlapping each copy’s T with its neighbour’s T', to form a strip. Moreover, in general we can apply any number of substitution steps to Q to obtain larger patches that repeat to form wider strips. This approach is used in the examples in the following section.

Translationally repetitive tilings are common, providing abundant source material for constructing periodic strips. For example, substitution rules normally produce repetitive tilings, and I am not aware of any substitution tilings that are not also translationally repetitive. Indeed, the Radin and Wolff result can be restricted to translation, so that if a set of shapes admits a tiling, the set must admit tilings that are translationally repetitive.

Examples

In this section I show a variety of examples of periodic strips, to demonstrate the generality of the construction and the expressive range they permit.

The chair rep-tile. I begin with the chair tile [3, Section 10.1] shown in Figure 3. The chair is a 4-rep-tile: four copies of the tile may be arranged to create a scaled copy of the original (Figure 3, left). When this rule is iterated it produces larger and larger patches, which in the limit define an attractive non-periodic tiling of the plane. The chair's substitution rule produces a patch containing two chairs in the same orientation, shown shaded in the figure. We can therefore use the bottom-up construction from the previous section, defining Q from those two tiles. Applying zero, one, or two rounds of substitution and placing overlapping patches along a line produces the three friezes shown in Figure 3, right.

The constructions in this paper are overkill for the chair, given that it also tiles periodically. In fact, the friezes shown here already appear as-is in the tiling produced via substitution. Still, the chair serves as an accessible first example, and shows that true aperiodicity is not a requirement for these constructions.

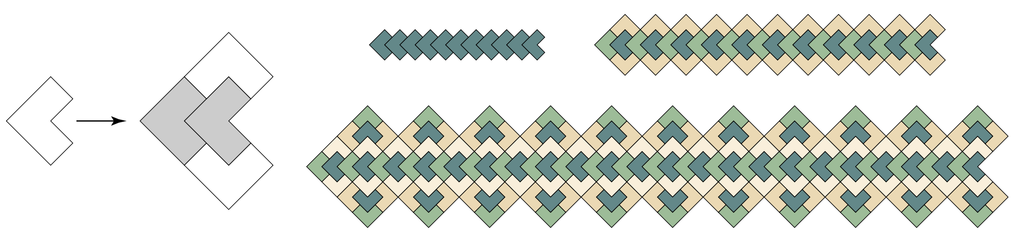

Figure 3: *The chair rep-tile. A substitution rule (left) expresses a scaled-up chair as the union of four chairs. The shaded chairs can serve as the basis for constructing periodic strips (right).*

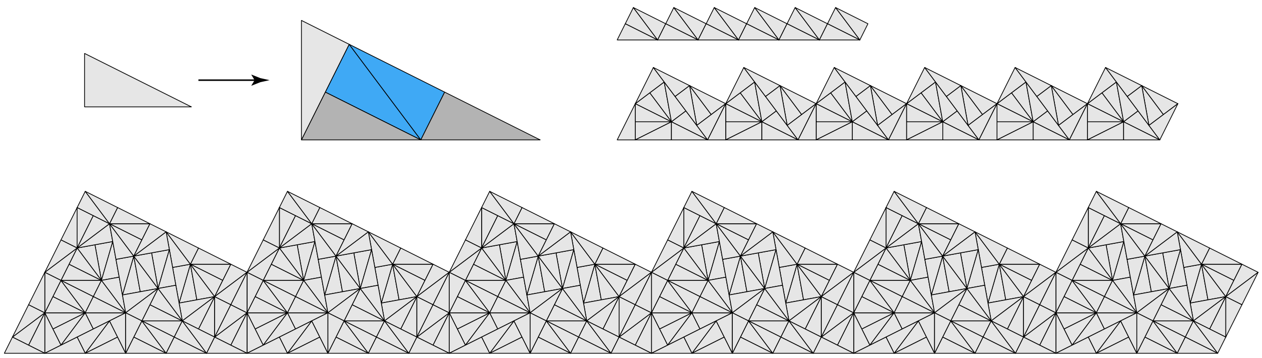

Figure 4: *The substitution rule in the upper left defines the pinwheel tiling. The dark grey and blue triangles can be used as the basis for constructing periodic strips of different widths (right and bottom).*

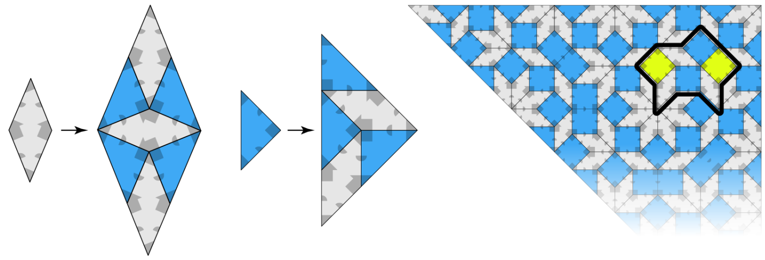

Figure 5: *The substitution rules for the Ammann-Beenker tiling (left). When iterated three times, these rules produce a patch containing two squares of the same orientation (highlighted in yellow), which can be used to define a sub-patch (outlined in bold) for constructing periodic strips.*

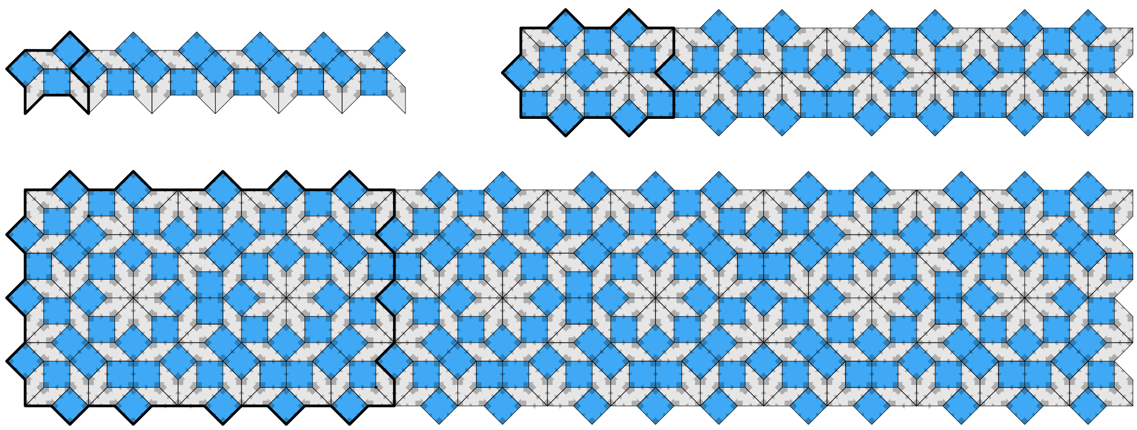

Figure 6: *Periodic strips constructed from the patch in Figure 5 after zero, one, or two substitution steps.*

The pinwheel tiling. The Conway-Radin pinwheel tiling [6] is a 5-rep-tile based on a right-angle triangle with side lengths 1, 2, and $\sqrt{5}$. Pinwheel tilings include tiles in infinitely many orientations. Nevertheless, the substitution rule produces two tiles in the same orientation, shown in dark grey in Figure 4. Every line segment connecting the interiors of those tiles passes through the two blue tiles. Here, then, the patch Q will contain four tiles. The figure shows overlapping copies of Q after zero, one, and two substitution steps. Although these strips repeat, they retain the somewhat chaotic appearance of the pinwheel tiling, which may be more interesting than simpler strips based on periodic tilings by this triangle.

The Ammann-Beenker tiling. The Ammann-Beenker tiling [3, Section 10.4] has two prototiles: a square and a 45° rhomb. It is a kind of analogue of Penrose rhombs, with fourfold local rotational symmetry rather than fivefold. The tiling can be constructed using substitution rules based on a rhomb and a half-square (Figure 5). The substitutions preserve a complex set of matching rules visualized by markings on the tiles and always pair half-squares into full squares.

To construct periodic strips I applied three rounds of substitution to a half-square, producing a patch in which two full squares and their markings appear in the same orientation for the first time. These squares are shown highlighted in yellow in Figure 5, right. The line segment connecting the centres of these squares comes into contact with all the other rhombs and squares contained within the bold outline, yielding a patch Q that can repeat. Figure 6 shows strips made from copies of Q after zero, one, and two substitutions. Note that the strips are all compatible with the markings that express the matching rules.

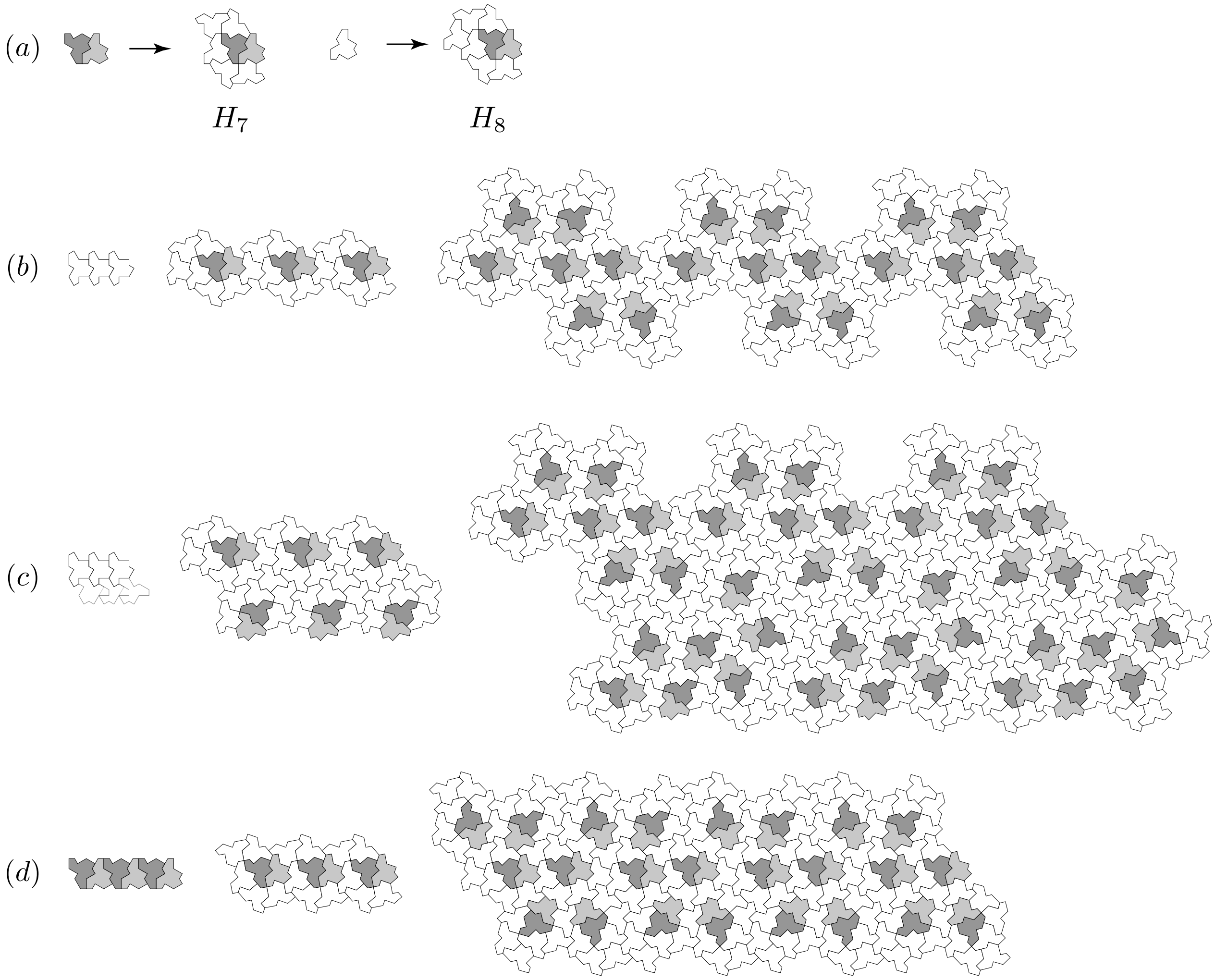

Figure 7: *Periodic strips based on the hat aperiodic monotile. The H7/H8 substitution rules (a) can be used to construct patches of tiles. A naive repetition that starts with a row of hats (b), to which the H8 rule applies, produces strips that do not grow ever wider. This problem can be avoided by starting with overlapping hats (c) or a row of two-hat compounds (d).*

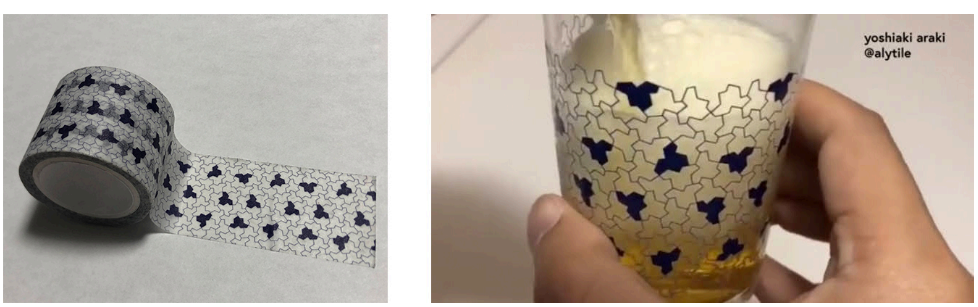

Figure 8: *A roll of transparent tape printed with a frieze of hats (left), applied to a drinking glass (right). Design and photographs by Yoshiaki Araki.*

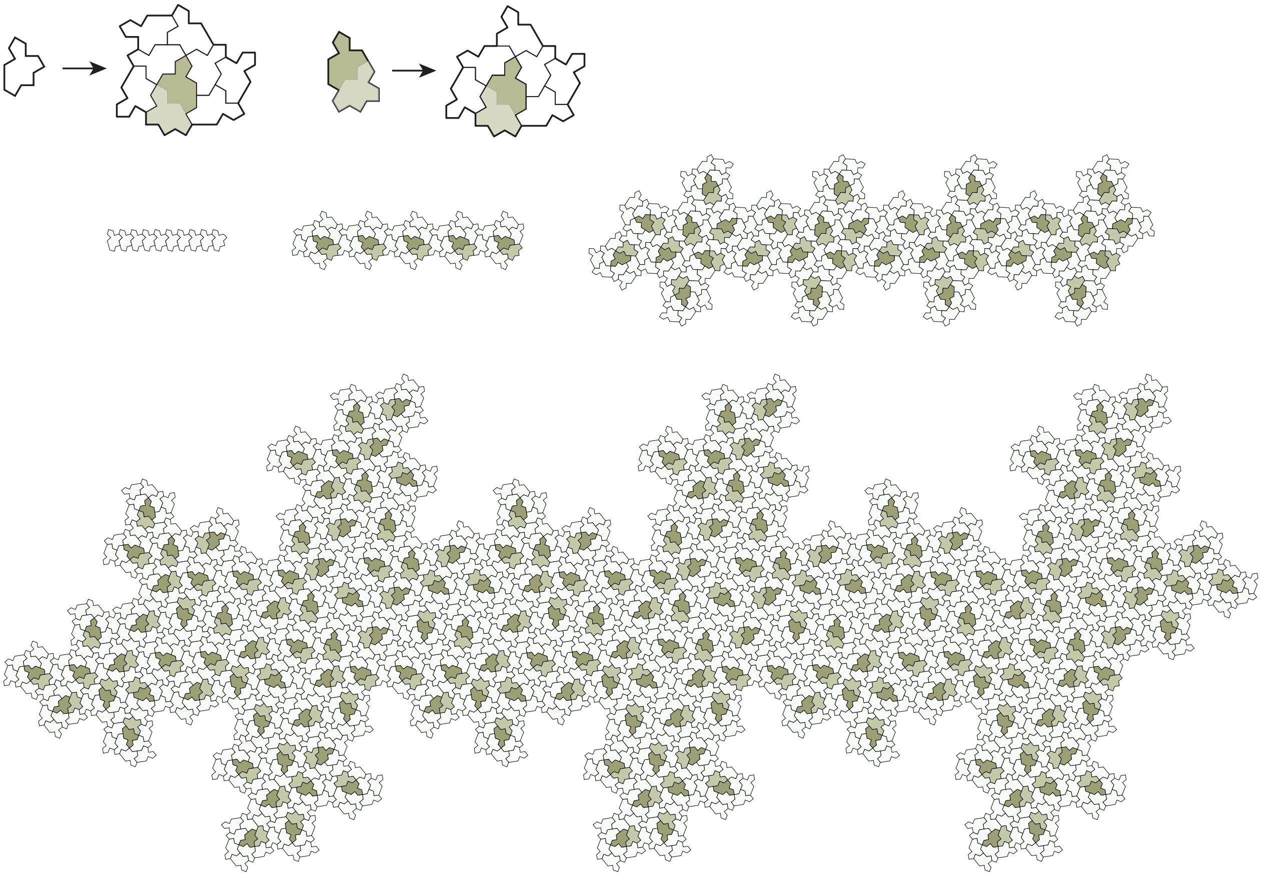

Figure 9: *Periodic strips based on substitution rules for Spectres and Mystics (top), and drawn using copies of* Tile(1, 1).

The hat. The hat [8] is a recently discovered aperiodic monotile: a shape that admits only non-periodic tilings, with no special matching rules needed. The "H7/H8" substitution rules [8, Figure 2.11], shown in Figure 7a, offer a convenient basis for constructing periodic strips, with some caveats. The H8 substitution rule produces a patch with two adjacent hats in the same orientation. However, if we repeatedly perform substitution on a row of hats stacked this way, the resulting strip acquires deep concavities that prevent it from getting wider (Figure 7b). One workaround is to start with a base patch Q that contains overlapping tiles (Figure 7c); the overlaps resolve after a single substitution. It is also possible to avoid this problem by starting with the two-hat cluster that drives the H7 rule (Figure 7d). Yoshiaki Araki has already used a periodic strip of hats as a decoration for a drinking glass (Figure 8).

The Spectre. Spectres are also aperiodic monotiles, discovered as an offshot of the hat, which admit tilings with tiles of uniform handedness [9]. The polygon known as Tile(1, 1) admits equivalent tilings if we artificially prohibit the mixture of left- and right-handed tiles in the same patch. These tilings can be defined using a substitution system on tiles consisting of a single tile and a two-tile cluster called a Mystic (Figure 9, top). As with the hat, these tilings contain adjacent copies of Tile(1, 1) in the same orientation, which can be used to construct periodic strips (Figure 9).

Discussion

The existence of periodic strips of any width for aperiodic tile sets is mathematically fascinating. Grünbaum and Shephard proved that, under relatively mild conditions, if a tile set admits a tiling with any translation symmetry, then it must also admit a periodic tiling [3, Theorem 3.7.1]. At first glance it would seem that if we can construct periodic strips of any finite width, then in the limit as width increases we obtain a tiling of the plane with global frieze symmetry, from which we can deduce the existence of a periodic tiling, even for an aperiodic tile set! The need to avoid this apparent contradiction allows us to conclude that as we build strips of increasing widths, their periods must also grow without bound. That way, in the limit the period goes to infinity along with the width, at which point the illusion of a tiling with frieze symmetry evaporates.

The possibility of constructing periodic strips from aperiodic tile sets is not entirely new. Kimberly-Clark's ill-fated toilet paper experiment provides real-world evidence of this fact. Furthermore, the existence of periodic strips of all widths is implicit in the construction used by Dworkin and Shieh to prove the existence of their "deceptions" [1]. The present paper provides a useful addition to the literature by highlighting periodic strips in isolation, giving a few simple ways to construct them, and demonstrating the process with many examples.

Acknowledgements

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