Labyrinths and Space-Filling Curves, Spirals and Tessellations: Topological and Geometrical Implications of Cartesian to Polar Transformations

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Abstract

Cartesian to polar transformations that map a square into a disk are examined according to their topological and geometrical implications that highlight the link between classical labyrinths and space-filling curves and the emergence of spirals in rounded ancient Roman tessellated mosaics, and lead to more general reflections.

Cartesian to Polar Transformations

Cartesian to polar transformations have been considered in the search for "computer generated beautiful images" by Elliot [4], complex forms by Greenfield [5], or *motifs* by Bleicher [1]. They have even become a familiar tool, being present in image editing softwares, though I wrote my own tool in *Processing*. My intent is not to produce spectacular images but to reflect upon the particular topology and geometry of the resulting space by comparison with those of the initial space.

A Cartesian to polar transformation consists in taking the Cartesian coordinates (x,y) of any point in the plane, and considering them as polar coordinates (r,θ). For our purpose in this paper, we shall consider mapping a square into a disk, with the side of the square equaling the diameter d of the disk. To find the polar coordinates (r,θ) of the transformed point P' of point P with Cartesian coordinates (x,y), one can use the transformation: $r = x/2$ and $\theta = 2\pi y/d$ (Figure 1).

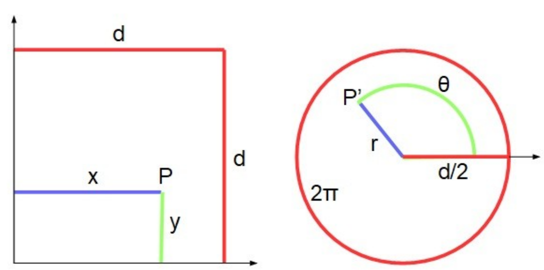
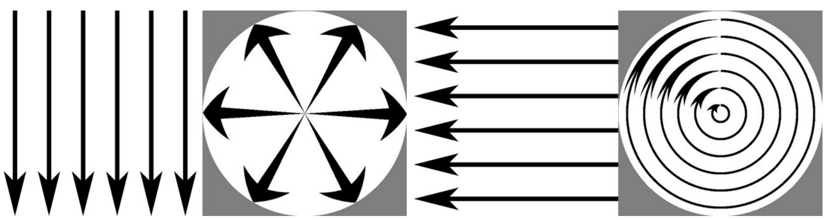


Figure 1: *Cartesian to polar transformation.*

There are other possibilities, by inverting the role of x and y or by changing the direction of the polar axis. In order to implement such a transformation as a computer graphics program, one must be aware that, while this transformation is a bijection from the square to the disk (except for those points that map into the center of the disk), as a transformation between pixels, it is not. So the only way to obtain a correct transformed image is to consider each pixel in the resulting disk and wonder from which pixel in the square it comes from. Also, in *Processing*, as in many other computer graphics systems, the y axis goes downwards. All of this considered, we can see what this transformation (with the polar axis upwards) does to a portrait of the creator of the Cartesian coordinates (Figure 2).

Figure 2: *Cartesian to polar transformation applied to a portrait of René Descartes (after Frans Hals, Le Louvre, Public domain).*

It has largely been observed that in such a transformation, lines are transformed into either radial lines or concentric circles. In our case, vertical lines are transformed into radial lines (Figure 3(a)), while horizontal ones are transformed into concentric circles (Figure 3(b)).



(a) (b)

Figure 3: *Cartesian to polar transformation of lines: (a) vertical, (b) horizontal.*

It reveals some fundamental topological facts. Our initial image is clearly defined as having a top and a bottom, a left and a right side. It is especially conspicuous in Figure 2 because it is a portrait. Our final image, however, is defined as having a center and a perimeter, a unique border, and no sides. Vertical lines are transformed into lines that start at the center and end at the border, and horizontal lines (or segments) are transformed into circles, which are closed loops. The topology of the resulting disk, or polar disk, is consequently different from that of the initial Cartesian square, and an intermediate step could be to consider the square with a cylindrical, or periodic, topology, i.e. where the left and right sides are supposed to meet. Let us add finally that the particular circular shape of the border of the disk is irrelevant for these topological considerations, and could be any closed curve.

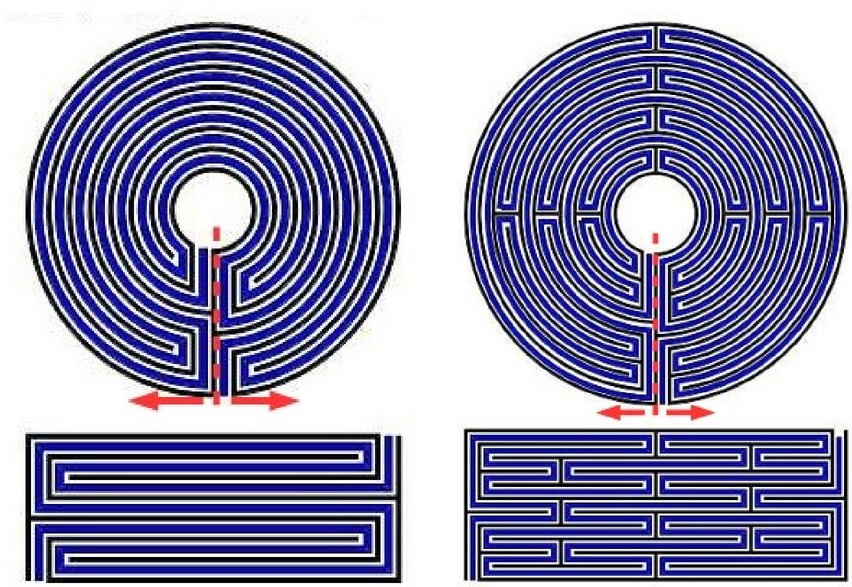
Labyrinths and Space-Filling Curves

My first use of such a transformation occurred through my investigation of labyrinths [2].

Contrary to a familiar connotation of the term, medieval and ancient Cretan labyrinths are unicursal, which means that there is only one path, without any branching or loop, and that it leads inevitably from the entry on the perimeter towards the arrival at the center (no need for Ariadne's thread…). The path of a labyrinth is delimited by walls, which can be true walls, or bushes, or differently colored tiles on the floor, or even simple lines. In the case of the famous digital labyrinth carved on a pillar of the portico of Lucca Cathedral, the "walls" are grooves that delimit the elevated path (Figure 4).

Figure 4: *The digital labyrinth on a pillar of the portico of Lucca Cathedral.*

Ancient labyrinths (as seen in Figure 5(a)) differ slightly from medieval ones, but the principle is the same. In any case there is always one wall going from the center to the perimeter. One can then cut through this wall, and "spread" the pattern (Figure 5).



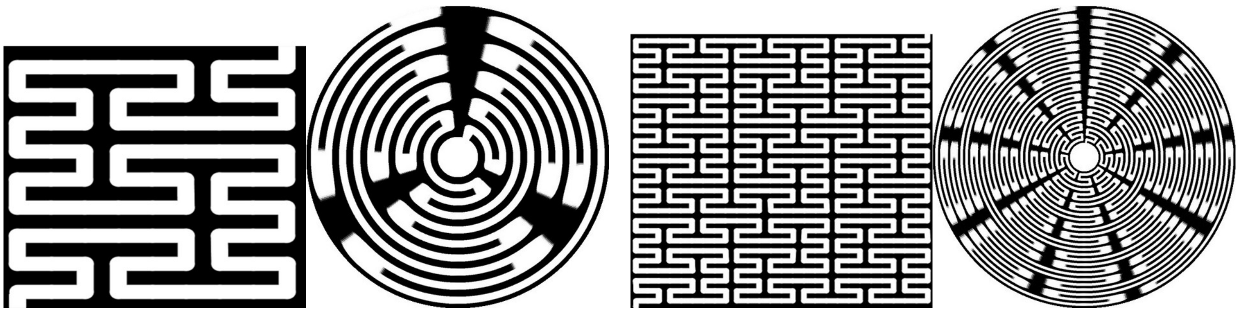
(a) (b)

Figure 5: *Spreading of labyrinths: (a) Cretan, (b) medieval.*

The path of a labyrinth, though unicursal, is not straightforward, it is meandering, contorted, complicated in the etymological sense, i.e. folded, and refolded into itself. It conveys the feeling that the goal is unreachable, though actually the arrival, the end is ineluctable. Moreover, the path visits every part of the domain in which the labyrinth is enclosed.

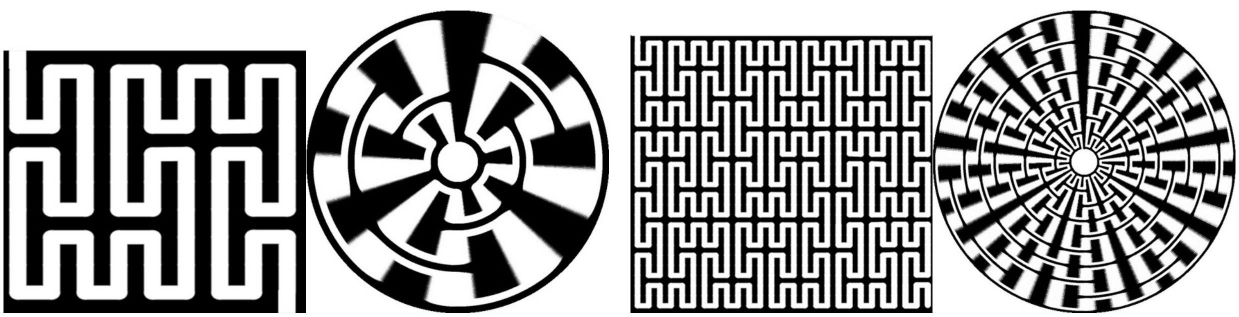
These features are close to those of space-filling or FASS (space-Filling, self-Avoiding, Simple and self-Similar) curves which are also unicursal, are recursively folded, and visit not only every part of a portion of the plane, but every point, till filling it completely. Arguably the most famous of these curves are the Peano and the Hilbert curves (see step 2 (Figure 6(a) and 7(a)) and 3 (Figure 6(b) and 7(b)) of the Peano curve and step 3 (Figure 8(a)) and 4 (Figure 8(b)) of the Hilbert curve, as L-systems, in white on black).

The main difference between labyrinths and FASS curves is that for the former the path goes from the perimeter to the center of a disk (or variants), while for the latter, it goes from one corner to another corner of a square. It is here that the Cartesian to polar transformation becomes useful. For the Peano curve, that starts and ends on opposite corners, there are two possible variants (Figure 6 and 7 resp.).



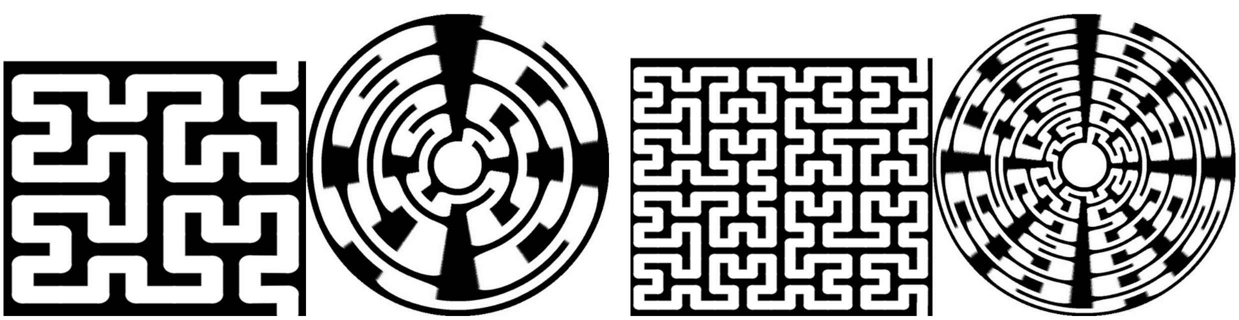
(a) (b)

Figure 6: *Cartesian to polar transformation of the Peano curve (variant #1): (a) step 2, (b) step 3.*



(a) (b)

Figure 7: *Cartesian to polar transformation of he Peano curve (variant #2): (a) step 2, (b) step 3.*



(a) (b)

Figure 8: *Cartesian to polar transformation of the Hilbert curve: (a) step 3, (b) step 4.*

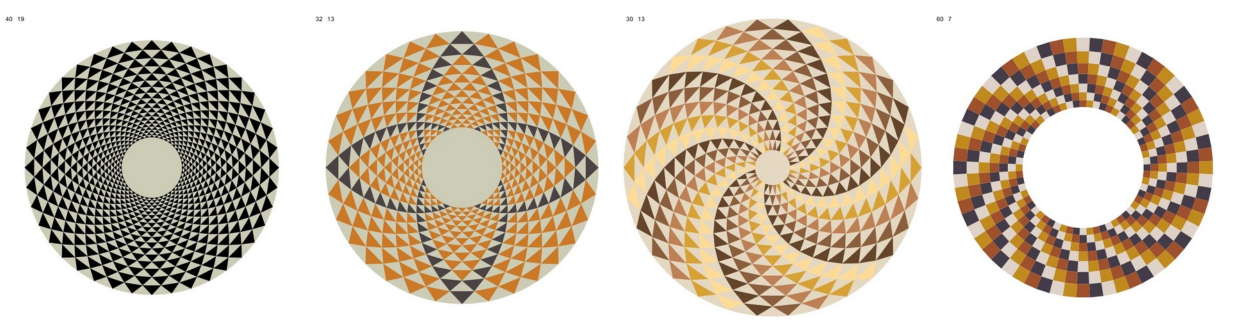
This experiment provides new labyrinths, and emphasizes the particular topology of the polar disk, giving clearly a different role for the center and the perimeter. But it does not take advantage of its periodic topology, since, just as classical labyrinths, there is one wall going from center to perimeter, and the path does not cross it. A curve that can do that, however, is the spiral, which we shall encounter in a very different context.

Spirals and Tessellations

My second encounter with the Cartesian to polar transformation occurred when analyzing ancient Roman mosaics in the form of disks, sometimes also called roundels [3].

It was while working on phyllotaxis that a figure in Jean [6], showing a bad reproduction of such a mosaic (Roman mosaic with Head of Medusa, 115-150 AD. Museo Nazionale Romano. Palazzo Massimo alle Terme.), drew my attention. This image was weird because, while obviously showing spirals, those were absolutely not phyllotactic spirals. This mosaic is composed with a constant number of triangular tiles, arranged in concentric circles, or rows, around a smaller decorated disk. The conspicuous spirals are of the same number as that of the triangles in each row, and there are the same number of clockwise and anticlockwise spirals, all characteristics alien to phyllotaxy.

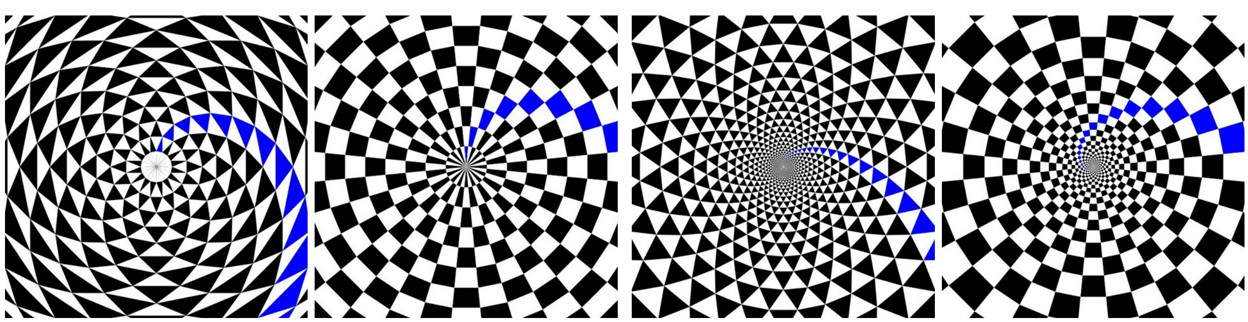
There are actually a few mosaics of the same type. Most are made with triangular tiles, either with dark triangles on a light background (Figure 9(a)), or with colors enhancing the spirals (Figure 9(b) (c)). Some display weirder shapes, but that are part of triangles, very few display quadrilateral tiles (Figure 9(d)). I could not find any mosaics with hexagonal tiles. I first analyzed those patterns by counting the rows, and the shapes by row, and acknowledging the variations in size of the shapes. There are a majority of patterns with roughly isometric triangles, and consequently an increasing depth of the rows from center to perimeter (Figure 8 (a) (b)), but in one case at least, the mosaicist tended to keep this depth constant, and then got pointy triangles near the center, and more flattened ones near the perimeter (Figure 9(c)).



(a) (b) (c) (d)

Figure 9: *Analysis and simulation of: (a) Mosaic Floor with Head of Medusa, 115-150 AD. J. Paul Getty Museum, (b) Head of Dionysos in spiral pattern mosaic. Corinth, Greece, (c) Roman geometric mosaic roundel, circa 3rd Century AD, (d) Roman mosaic, Syria, circa 4th-5th century AD.*

Those patterns are obviously tessellations of the disk, and we can generalize them by producing some such tessellations without the void disk at the center, and extend them beyond the perimeter of a disk, with either constant or decreasing depth of the rows (or height of the shapes).



(a) (b) (c) (d)

Figure 10: *Generalized tessellations: (a) triangles, constant height, (a) quadrilaterals, constant height, (a) triangles, decreasing height,(a) quadrilaterals, decreasing height.*

The first tessellations with a constant depth (Figure 10(a) (b)) may also be interpreted as classical tessellations of the plane transformed by the Cartesian to polar transformation (Figure 11; we refer the triangular tessellations with the number of black tiles).

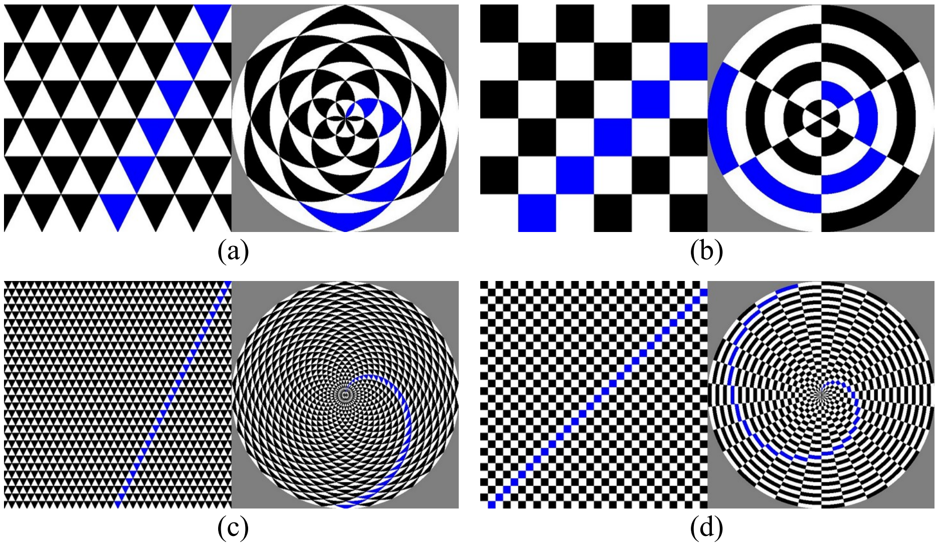


Figure 11: *Cartesian to polar transformation of tessellations:*
(a) 6 x 6 triangles, (b) 6 x 6 squares,(c) 30 x 30 triangles, (d) 30 x 30 squares.

Oblique lines produced by shifted tiles in classical Cartesian tessellations are not very conspicuous, though the spirals in their transformations are, at least when there is enough density (Figure 11 (c) (d)). And clever mosaicists enhanced them with their coloring.

This interpretation leads us back to our Cartesian to polar transformation and its implications. We saw what happened to horizontal and vertical lines. But what about oblique lines (Figure 12)?

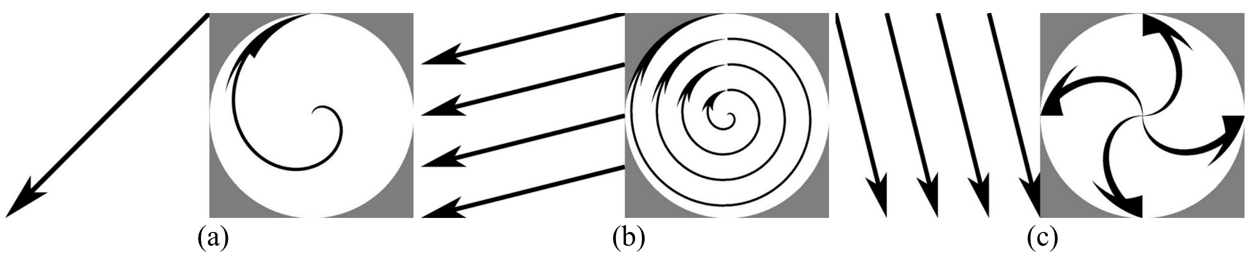


Figure 12: *Cartesian to polar transformation of oblique lines:*
(a) diagonal, regularly spaced, acute angle, (c) regularly spaced, obtuse angle.

We see that oblique lines are transformed into Archimedean spirals, either one spiral or multiple spirals. Now we can add a geometric characteristics to the polar disk: Archimedean spirals are as inherent to it as oblique lines are to the Cartesian plane. In Figure 12(a) and (c) the oblique lines going from top to bottom are transformed into spirals going from center to border (as did the path of the labyrinth in the first section). The case in Figure 12 (b) is particularly interesting: it exploits the periodic topology and transforms multiple oblique lines into a unique continuous spiral.

The other tessellations with a varying depth (Figure 10(c) (d)) lead us to introduce a new transformation where $\theta = 2\pi y/d$ as before, but now $r = a^x$, a being chosen conveniently for our disk to be of the right size. Let us call it a logarithmic (by analogy with such spirals) Cartesian to polar transformation and look at what it does to the portrait of Descartes (Figure 13), and to the tessellations (Figure 14):

Figure 13: *Logarithmic Cartesian to polar transformation applied to a portrait of René Descartes.*

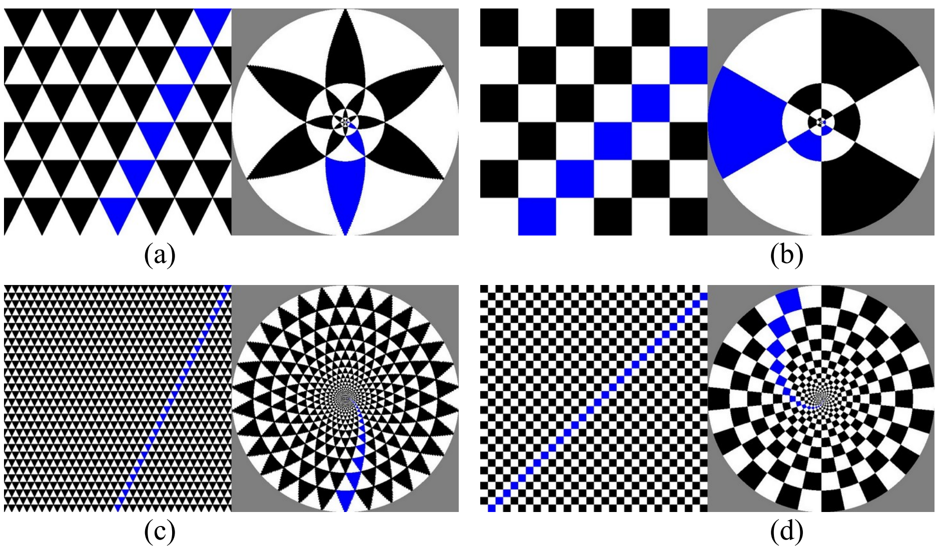


Figure 14: *Logarithmic Cartesian to polar transformation of tessellations:*
(a) 6 x 6 triangles, (b) 6 x 6 squares,(c) 30 x 30 triangles, (d) 30 x 30 squares.

And, finally, vertical lines are still transformed into radial lines, horizontal ones into concentric circles (Figure 15), and oblique lines into logarithmic spirals (Figure 16).

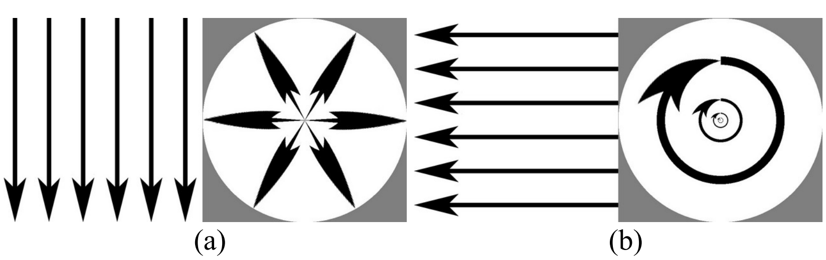


Figure 15: *Logarithmic Cartesian to polar transformation of: (a) vertical, (b) horizontal.*

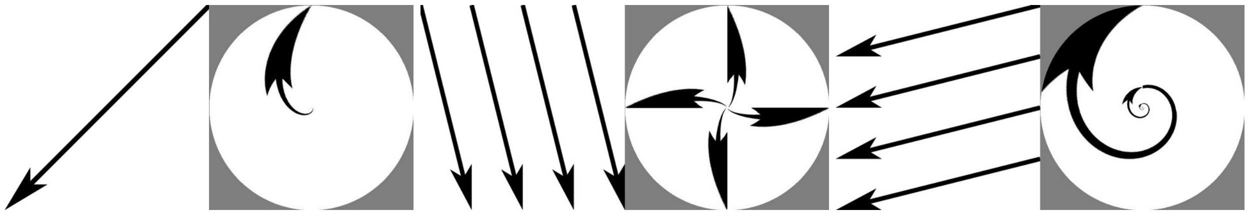


Figure 16: *Logarithmic Cartesian to polar transformation of oblique lines: (a) diagonal, regularly spaced, acute angle, (c) regularly spaced, obtuse angle.*

We have defined two Cartesian to polar transformations, mapping oblique lines into Archimedean spirals for the first one, and into logarithmic spirals for the second one. This could be prolonged by defining other Cartesian to polar transformations in which oblique lines are transformed into any other type of spirals.

Summary and Conclusions

Our journey into Cartesian to polar transformations has emphasized the particular topology of the polar disk, as having a center and one border but no side, and the emergence of spirals. Let us add again that for the topology, the particular shape of the border does not actually matter: a square can be considered as well, if it is viewed as having a center and a perimeter and no side. Labyrinths are often inscribed in a square, or another regular polygon. Our research originated in tessellated disk mosaics and labyrinths (which happen to occur in mosaics too) but it goes obviously beyond that, especially in architecture. The labyrinth is one of the origin myths of architecture. Defining a path is a main task for architects, and the route through a museum is an obvious example of such an endeavor. This question links our two topics. Two great architects chose the spiral, though inscribed in a square for Le Corbusier and his *Musée à croissance illimitée*, or its extension, the helix, for Frank Lloyd Wright and his *Guggenheim Museum*, but explorations of a more labyrinthine path have been made for instance by Peter Eisenman for the *Guangdon Museum*. Those examples show that, while labyrinths and spirals are two-dimensional patterns, they can be extended to three dimensions.

References

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