Supplementary Material for
Domain-Specific Languages for Efficient Composition of Paths in 3D

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More Images

We provide some additional illustrations for the example path expression used in the paper:

```
A:-|>
B:A>a>
B3
```

Figure 1 shows four stages of building up the path, as well as the full path using the same coloring.

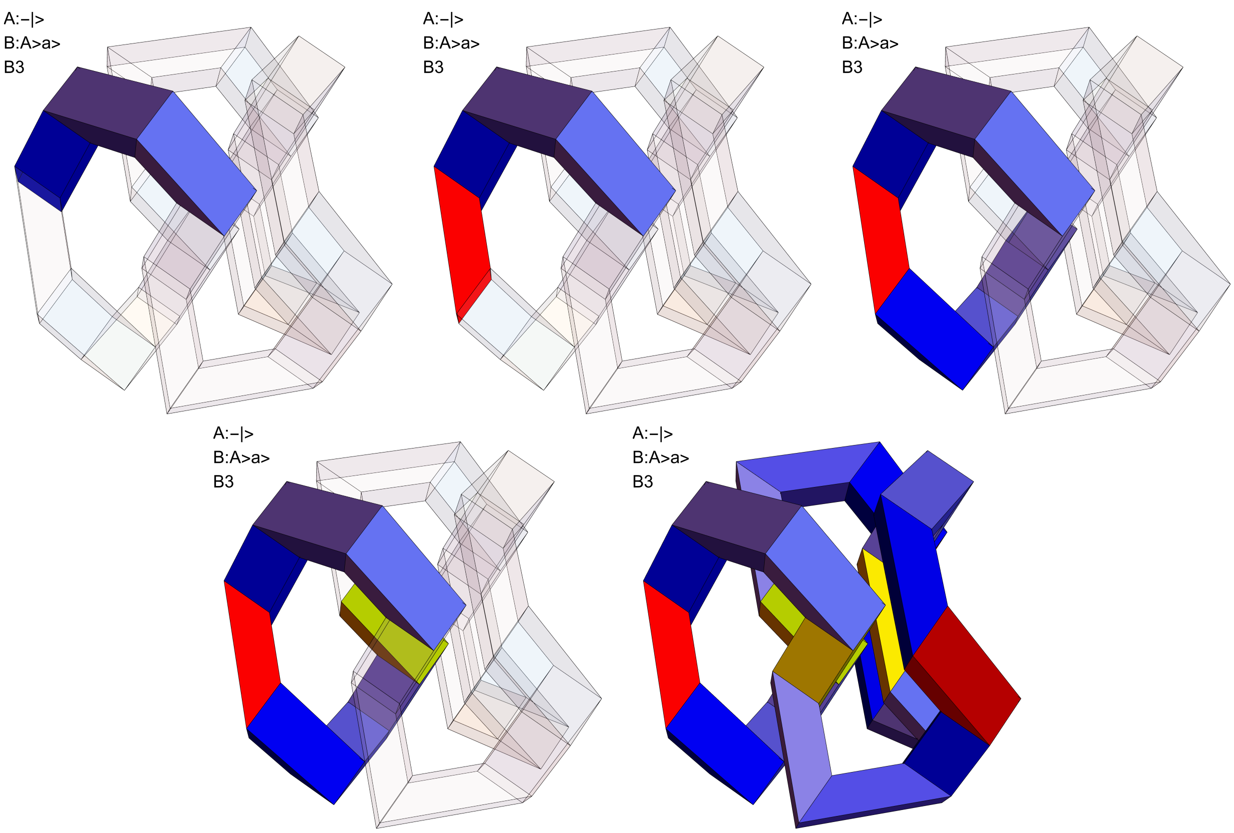


Figure 1: *Subpaths highlighted for* `A:-|>`, `A>`*, and* `A>a` *(in first row), and* `B:A>a>` *and* `B3` *(in second row).*

Figure 2 shows another FCC path for the example path expression. When all joint angles are constrained to |> (right or obtuse), excluding = (straight), and the travel distance is limited to 13%, there are only two paths generated. These are Figure 1 (also shown in the paper) and Figure 2 here.

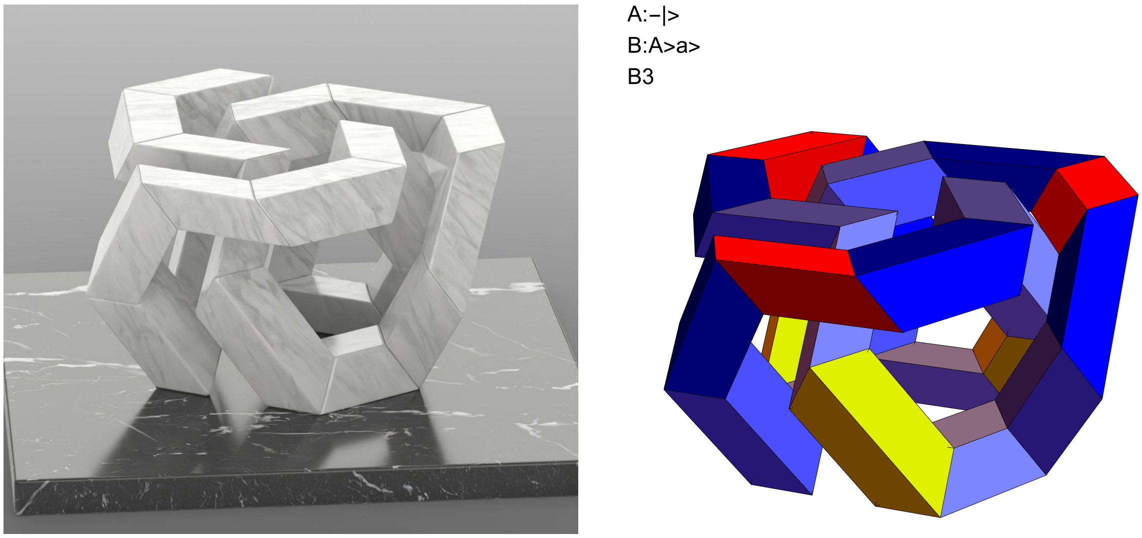


Figure 2: *Another solution to the example path expression (left: in marble, computer generated image; right: with subpaths highlighted).*

Figure 3 shows a piecewise linear rendering of Opus 185131, the knotted FCC spiral. Here the match with the path expression is easier to make:

```
A:--1.2_1.2_1.3_1.3_1.3_1.4_1.4_1.4_1.5
AA%
```

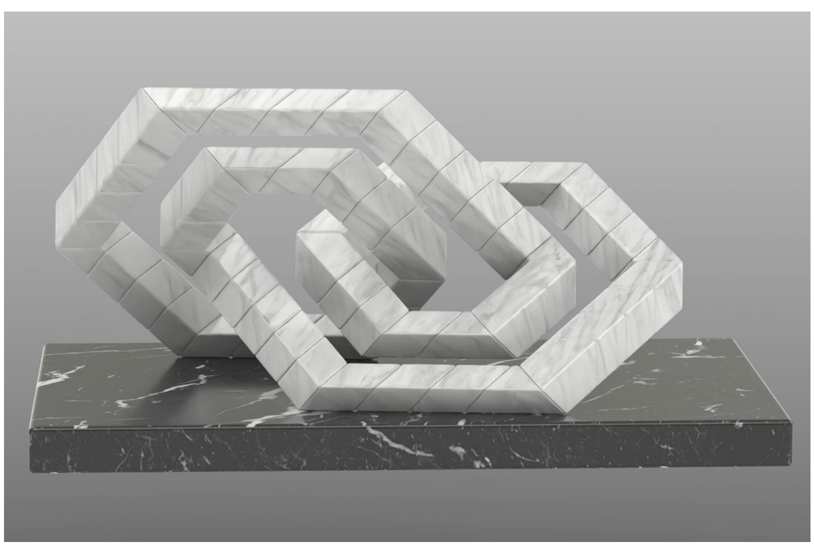


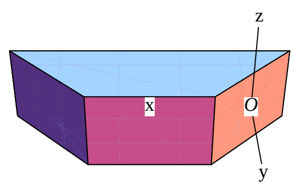
Figure 3: *Opus 185131 (piecewise linear): in marble (computer generated image).*

Other Turtle Geometry Specializations

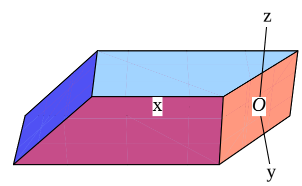
In [2], the following four abbreviations are introduced

$$
\begin{aligned}
T_d &= \mathit{Move}(d);\mathit{Turn}(90^\circ) && (1)\\
R_d &= \mathit{Move}(d);\mathit{Roll}(\ \ 90^\circ);\mathit{Turn}(90^\circ) && (2)\\
L_d &= \mathit{Move}(d);\mathit{Roll}(-90^\circ);\mathit{Turn}(90^\circ) && (3)\\
P_d &= \mathit{Move}(d);\mathit{Roll}(180^\circ);\mathit{Turn}(90^\circ) && (4)
\end{aligned}
$$

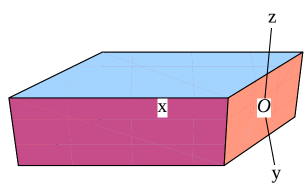
The names were chosen because they have a mnemonic value, when using bevelled square beams and miter joints to construct the path (see Figure 4). Using this DSL, all paths in the Simple Cubic lattice can be described as sequences of T, R, L, and P steps. For instance, Opus 951465 (Figure 2 in the paper), can be



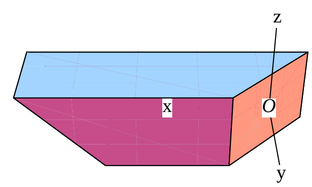
T_d has a trapezoid as projection



R_d makes right-handed spirals (mirror image of L_d)



P_d has a parallelogram as projection



L_d makes left-handed spirals (mirror image of R_d)

Figure 4: *Shapes of square beams bevelled for miter joints*

described by the following expression, where exponentiation indicates repetition:

$$\Big(L_1; R_5; (R_6)^2; L_3 \ \ ; \ \ R_1; L_5; (L_6)^2; R_3 \Big)^3 \qquad (5)$$

making 30 steps: a three-stranded right-hand spiral of 5 segments per strand ($L_1; R_5; R_6^2; L_3$), interwoven with a three-stranded left-hand spiral also of 5 segments per strand ($R_1; L_5; L_6^2; R_3$). It is congruent to its mirror image, that is, turning it upside-down is equivalent to reflecting it. This description can be directly matched to the path expression given in the paper, by taking $A = L_1; R_5; (R_6)^2; L_3$:

```
A:--1.5-1.6-1.6-1.3
B:AA%
B3
```

Also in [2], the following abbreviations are defined:

$$
\begin{aligned}
R'_d &= \mathit{Move}(d);\mathit{Roll}(\ \ 60^\circ);\mathit{Turn}(\arccos 1/3)\\
L'_d &= \mathit{Move}(d);\mathit{Roll}(-60^\circ);\mathit{Turn}(\arccos 1/3)
\end{aligned}
$$

where arccos $1/3 \approx 70.5°$. This language is used to describe BCC paths with obtuse angles. These are nicely realized with beams having an equilateral triangle as cross section. Opus 125707 (Figure 4 in the paper) can be described by

$$\left(L_1'; \left(R_2'\right)^2; L_1'; R_2' \;\; ; \;\; L_2'; R_1'; \left(L_2'\right)^2; R_1' \right)^2 \tag{6}$$

making 32 steps. This description can be directly matched to the path expression given in the paper, by taking $A = L_1'; \left(R_2'\right)^2; L_1'; R_2'$:

```
A:-8
B:Aa
BB
```

These notations are precursors of Anton's Path Language.

In [4], a DSL is defined for 3D paths based on two global constants, viz. the turn angle ϕ and the roll angle ψ and the following abbreviations:

$$\begin{aligned} + &= \mathit{Move}(1); \mathit{Roll}(+\psi); \mathit{Turn}(\phi) \\ - &= \mathit{Move}(1); \mathit{Roll}(-\psi); \mathit{Turn}(\phi) \end{aligned}$$

Koos Verhoeff typically fixed $\psi = 90°$ because that corresponds to the rotational symmetry of a beam with a square cross section. Paths described in this DSL are usually not lattice paths, and it is not obvious for which turn angles ϕ such a path is properly closed (also see [1][3]).

Finally, [5] defines a DSL to describe obtetrahedrille constructions. The turtle then travels in a sublattice of FCC. The DSL has a history mechanism to allow for branching.

More Information about the Cubic Lattices

Labels:

- O: origin (current point)
- P: previous point
- S: straight
- A through K: other next points (below, generically referred to as Q)
- A' through K', and P': pre-previous points (below, generically referred to as Q')

Measuring angles:

- Turn angles ϕ are measured when moving $P \to O \to Q$, such that $0 \le \phi < 180°$.

  In general, turn angle $P \to O \to S$ is 0.

- Roll angles ψ are measured when moving $Q' \to P \to O \to Q$, such that $-180° < \psi \le 180°$. The roll angle is the (directed) angle from the normal vector of the plane spanned by $Q' \to P \to O$ to the normal vector of the plane spanned by $P \to O \to Q$. The normal vector is determined via the right-hand rule; that is, for the first plane it equals $(P - Q') \times (O - P)$.

Simple Cubic (SC)

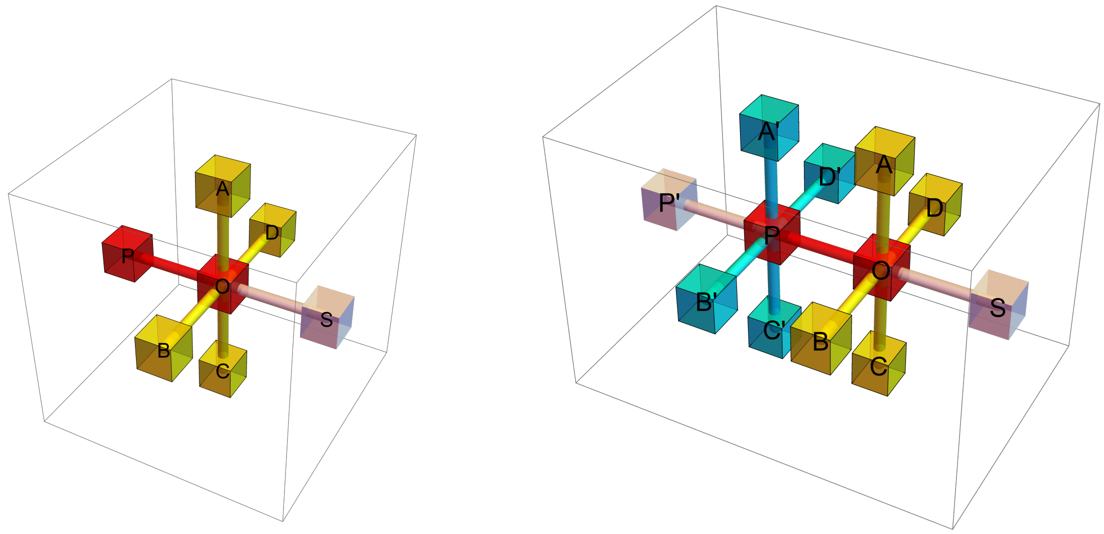


Figure 5: *Simple cubic lattice, some points and directions*

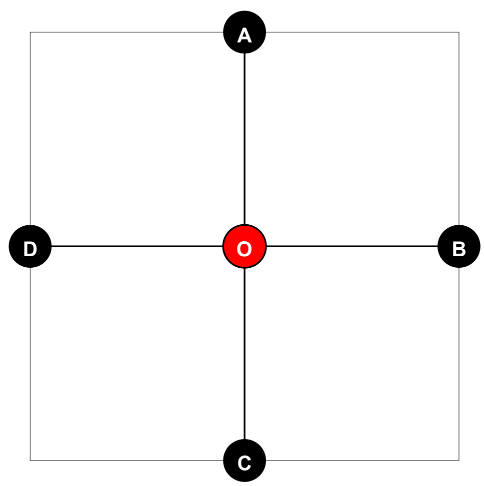


Figure 6: *Simple cubic lattice, projected in direction of PO, showing torsion angles*

- Number of directions: 5 (unconstrained), 4 (right angle)
- Turn angles: 0 (straight: POS), 90° (POA, POB, POC, POD)
- Roll angles: 0, ±90°, 180°

Face-Centered Cubic (FCC)

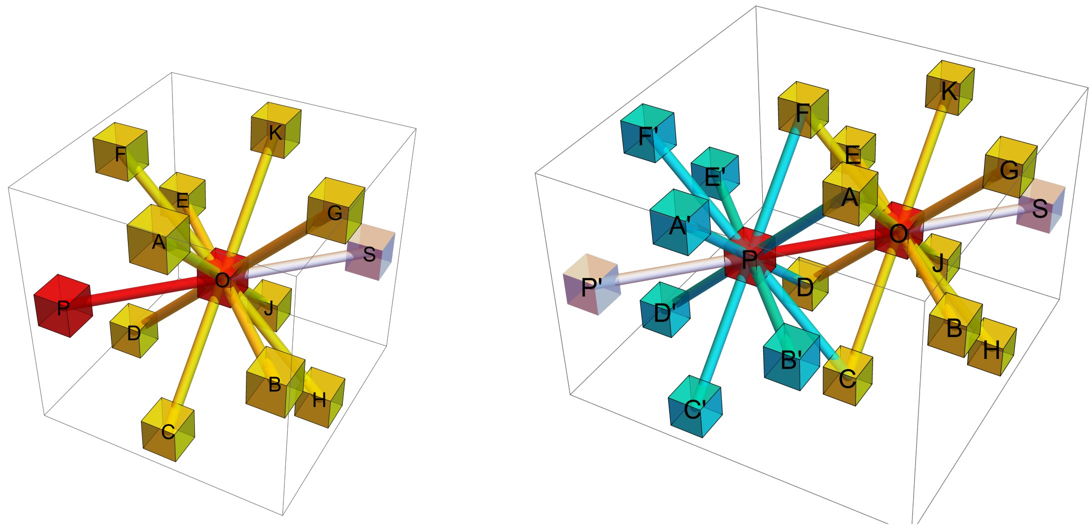


Figure 7: *Face-centered cubic lattice, some points and directions*

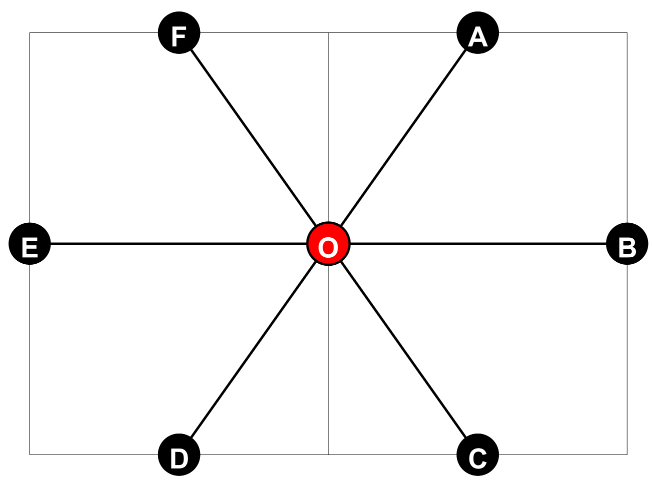


Figure 8: *Face-centered cubic lattice, projected in direction of PO, showing torsion angles*

- Number of directions: 11 (unconstrained), 4 (acute), 2 (right angle), 5 (obtuse, incl. 1 straight)
- Turn angles: 0 (straight: POS), 60° (POG, POH, POJ, POK), 90° (POB, POE), 120° (POA, POC, POD, POF)
- Roll angles: 0, $\arctan\sqrt{2} \approx 54.7356°$ (e.g. $A'POB$), $\arccos 1/3 \approx 70.5288°$ (e.g. $A'POF$), and supplements

Body-Centered Cubic (BCC)

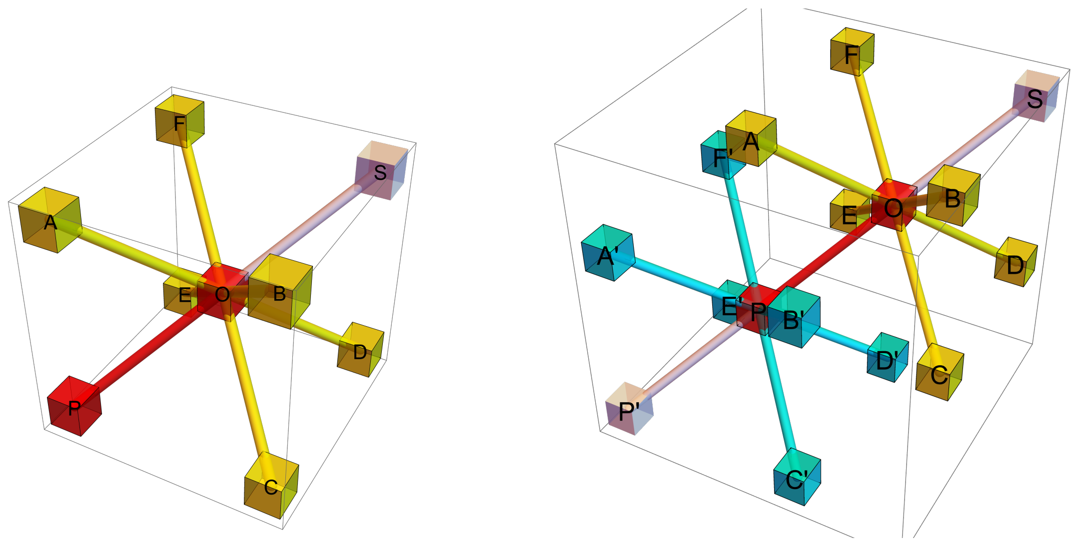


Figure 9: *Body-centered cubic lattice, some points and directions*

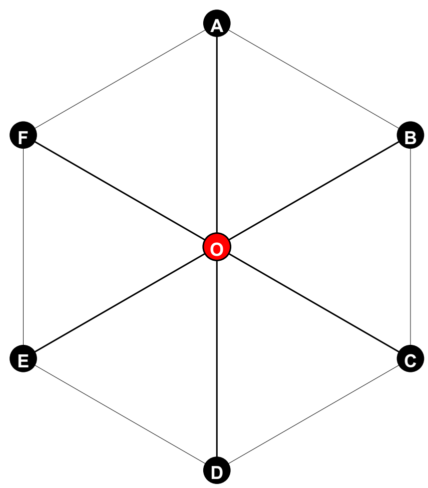


Figure 10: *Body-centered cubic lattice, projected in direction of PO, showing torsion angles*

- Number of directions (incl. straight): 7 (unconstrained), 3 (acute), 4 (obtuse, incl. straight)
- Turn angles: 0 (straight: POS), $\arccos 1/3 \approx 70.5288°$ (POB, POD, POF), $\arccos 1/\sqrt{3} \approx 109.471°$ (POA, POC, POE)
- Roll angles: 0, $\pm 60°$, $\pm 120°$

References

[1] M. van Veenendaal and T. Verhoeff. "Pretty 3D Polygons: Exploration and Proofs." *Proceedings of Bridges 2021: Mathematics, Art, Music, Architecture, Culture*. D. Swart, F. Farris, and E. Torrence, Eds. Phoenix, Arizona: Tessellations Publishing, 2021. pp. 111–118. http://archive.bridgesmathart.org/2021/bridges2021-111.html.

[2] T. Verhoeff. "3D Turtle Geometry: Artwork, Theory, Program Equivalence and Symmetry." *Int. J. of Arts and Technology*, vol. 3, no. 2/3, 2010, pp. 288–319.

[3] T. Verhoeff. "The Looping Theorem in 3D Turtle Geometry." *Proceedings of Bridges 2023: Mathematics, Art, Music, Architecture, Culture*. In these proceedings. Phoenix, Arizona: Tessellations Publishing, 2023.

[4] T. Verhoeff and K. Verhoeff. "Regular 3D Polygonal Circuits of Constant Torsion." *Bridges Conference Proceedings*. Banff, Canada, Jul. 26–30, 2009. pp. 223–230. http://archive.bridgesmathart.org/2009/bridges2009-223.html.

[5] T. Verhoeff and K. Verhoeff. "The Obtetrahedrille as a Modular Building Block for 3D Mathematical Art." *Proceedings of Bridges 2019: Mathematics, Art, Music, Architecture, Education, Culture*. S. Goldstine, D. McKenna, and K. Fenyvesi, Eds. Phoenix, Arizona: Tessellations Publishing, 2019. pp. 407–410. http://archive.bridgesmathart.org/2019/bridges2019-407.html.