# Topological Interlocking, Truchet Tiles and Self-Assemblies: A Construction-Kit for Civil Engineering Design

Reymond Akpanya[1], Tom Goertzen[2], Sebastian Wiesenhuetter[3],
Alice C. Niemeyer[4] and Jörg Noennig[5]

[1] Department of Mathematics, RWTH Aachen; akpanya@art.rwth-aachen.de
[2] Department of Mathematics, RWTH Aachen; goertzen@art.rwth-aachen.de
[3]Faculty of Architecture, TU Dresden; sebastian.wiesenhuetter@tu-dresden.de
[4] Department of Mathematics, RWTH Aachen; alice.niemeyer@art.rwth-aachen.de
[5] Faculty of Architecture, TU Dresden; joerg.noennig@tu-dresden.de

## Abstract

Topological interlocking is an abstract concept which requires that, given an assembly of blocks with a fixed frame as a constraint, no group of blocks can be removed. We introduce a construction kit based on a block, called Versatile Block, that leads to a wide range of possible topological interlocking assemblies. We show that the combinatorics of assembling copies of this block can be linked to Truchet tiles. We analyze the self assembly behaviour of the block by presenting experimental results based on 3D-printing with inserted magnets and computational experiments using the computer algebra system GAP. Furthermore, we present structural examples demonstrating the functionalities of each block in the construction kit. We investigate different design strategies and discuss possible applications in civil engineering design, considering possible manufacturing with different choices of material such as carbon reinforced concrete.

## Introduction

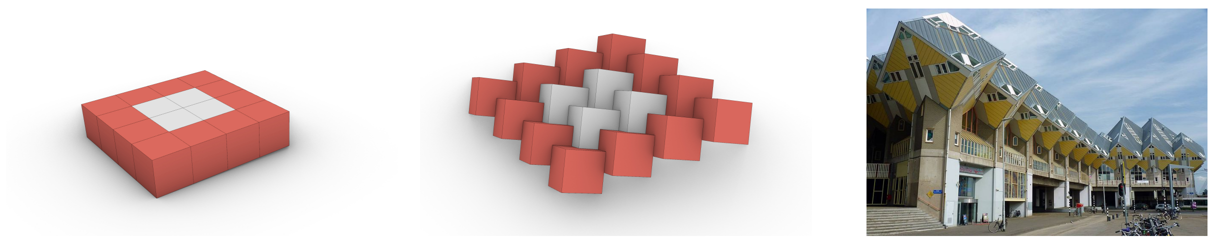
The building sector is facing diverse sustainability challenges, see e.g. [14]. Employing modular building blocks is an approach to tackle the challenge of reducing greenhouse gas emissions: design complexity is reduced to single blocks which can be recycled by reassembling structures, and production is simplified to single components. An arrangement of blocks that are in contact with each other is said to be *topologically interlocked* (or *TI*) if these blocks can be surrounded by a fixed frame such that no subset of blocks can be moved. Ensuring that blocks are topologically interlocked is particularly useful in a modular approach as local stresses are distributed globally, see e.g. [5, 6].

Additionally, saving material and developing materials with increased durability are important factors for sustainability. The collaborative research cluster CRC/TRR 280 [2] aims to provide innovative design strategies for minimal material solutions with inspiration and methods from areas as diverse as mathematics, biology, arts, architecture and civil engineering, see e.g. [20]. Topological interlocking is often used in compression-loaded-structures such as in dry-stone walls or in masonry, see e.g. [12]. Applying TI to new shapes manufactured from modern materials and by new fabrication methods has the potential to create innovative and sustainable design strategies. For instance, 3D-printing and robotics can be used to manufacture interlocking blocks with carbon reinforced concrete (CRC), see e.g. [13].

We present a construction-kit of blocks based on a block introduced in [8] as a candidate for solving the problems mentioned above. We show that there exists a wide range of possible TI assemblies based on this kit and we present examples, discuss properties and highlight the versatility of this construction-kit. Using only the basic block, we show that the concept of Truchet tiles can be used to classify planar assemblies and describe ways to investigate and quantify self-assemblies. Another construction kit proposed in [15] is based on the osteomorphic block, first introduced in [11].

## Topological Interlocking Assemblies

The concept of TI was introduced by Dyskin et al. in [5]. In [3] the authors give a method for constructing TI assemblies of convex bodies, in particular for each platonic solid. Below we see an example of two assemblies of cubes with the frame consisting of the red (outer) blocks: one admitting a TI assembly and the other not. The assembly shown in Figure 1(b) gives an alternative way of assembling cubes and was used by Piet Blom in the design of the Cube House in Rotterdam; see Figure 1(c). In [8], the authors describe a method to exploit wall-paper symmetries to construct a wide-range of TI assemblies by deforming fundamental domains. For a recent overview of research in this field see [6].



(a) (b) (c)

**Figure 1:** *(a) Simple cube assembly (b) TI assembly [3] (c) Cube-House [4].*

In [8], the authors give a definition of a TI assembly. We give an abbreviated version below, where a block is a 3-dimensional manifold in $\mathbb{R}^3$.

**Definition 1** ([8])**.** *Given a family of blocks $(X_i)_{i\in I}$, where $X_i \subset \mathbb{R}^3$ and $I$ is an index set, we say that $(X_i)_{i\in I}$ is a topological interlocking assembly for a given frame of fixed blocks $J \subset I$, if the following holds:*

1. *$(X_i)_{i\in I}$ is an assembly, that is any two blocks can only intersect at their boundaries;*
2. *Any finite set of blocks $S \subset I \setminus J$ cannot be moved using continuous motions (interlocking property).*

The structural examples given in the following sections lead to candidates for TI assemblies by choosing a suitable frame; see for instance Figure 2(c). Proving the interlocking property can be done by computing all contact faces and showing that any kind of motion leads to an intersection of blocks. An infinitesimal criterion is given in [19], where the authors establish a linear optimization problem such that the interlocking criteria is equivalent to showing that a corresponding linear problem has no solution.

## Topological Interlocking based on the Versatile Block

In [8], a block is constructed that we call the *Versatile Block* by exploiting a set $G$ of wall-paper symmetries. The idea is to deform two fundamental domains of $G$ placed in parallel planes continuously into each other. A rectangle and a square appear both as fundamental domains of several wall-paper groups, for instance p1, pg and p4 in crystallographic notation. A rectangle with side lengths 1 and 2 has the same area as a square with side length $\sqrt{2}$. By placing the square and rectangle as mentioned above in parallel planes and adding triangular faces as described in [8], we obtain a block which can be assembled in many possible ways due to its universal nature, see Figures 2(a), 2(b). The list of coordinates in $\mathbb{R}^3$ for the nine vertices of the *Versatile Block* is given by: $[(0,0,0),(1,1,0),(2,0,0),(1,-1,0),(0,1,1),(1,1,1),(1,0,1),(1,-1,1),(0,-1,1)]$. We will refer to a particular vertex by its position in this ordered list. For the underlying incidence structure of the triangulation as seen in Figure 2(a) the faces are given by the following lists of incident vertices.

$$[[1,2,3],[1,2,5],[1,3,4],[1,4,9],[1,5,9],[2,3,7],[2,5,6],$$
$$[2,6,7],[3,4,7],[4,7,8],[4,8,9],[5,6,7],[5,7,9],[7,8,9]].$$

Up to isomorphism, there are six ways of assembling two copies of the Versatile Block, shown in Figure 2(b).

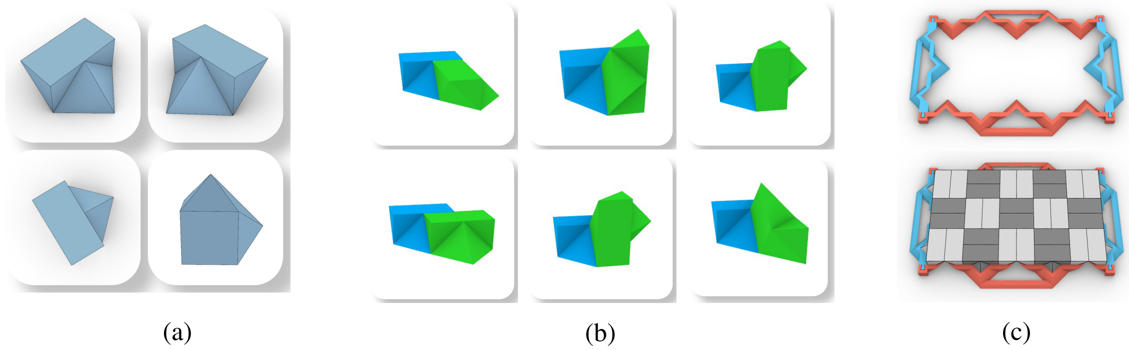



**Figure 2:** *(a) Various views of the Versatile Block, (b) All 6 possible assemblies of two copies of the Versatile Block, (c) Example of a frame for a TI assembly.*

Assembling two copies of the Versatile Block between the planes $z = 0$ and $z = 1$ given by $\langle(1,0,0),(0,1,0)\rangle$ and $\langle(1,0,0),(0,1,0)\rangle + (0,0,1)$ as shown in the left-most pictures in Figure 2(b), leads to a wide range of possible assemblies, called *planar assemblies*. A potential frame for such a TI assembly is given in 2(c). The planar TI assemblies can be classified by Truchet tiles (for example [17]) together with an assembly-rule as a combinatorial model. Each Truchet tile encodes a certain placement of the Versatile Block as given in the first four pictures of Figure 3, where we show the top and the bottom view of the corresponding position of the Versatile Block. The last two pictures show two combinations of Truchet tiles and the corresponding assemblies of two copies of the Versatile Block.

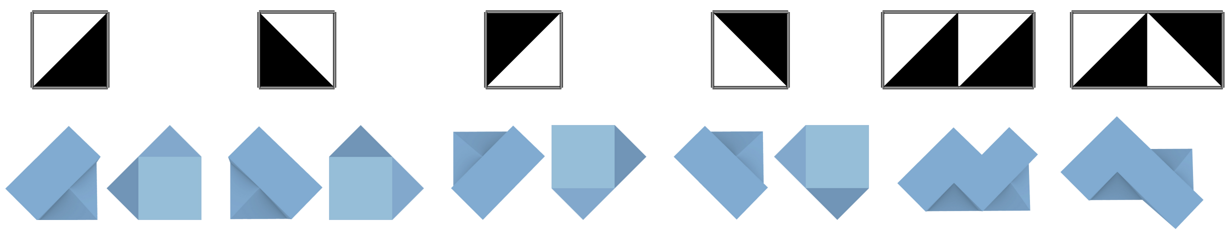


**Figure 3:** *Modelling the Versatile Block by Truchet tiles.*

For planar assemblies without holes, we can deduce the following rule from Figure 2(b) and Figure 3: Two different tiles can only touch at different colours. In Figure 4 we illustrate an example of possible tilings with corresponding Truchet tiles.

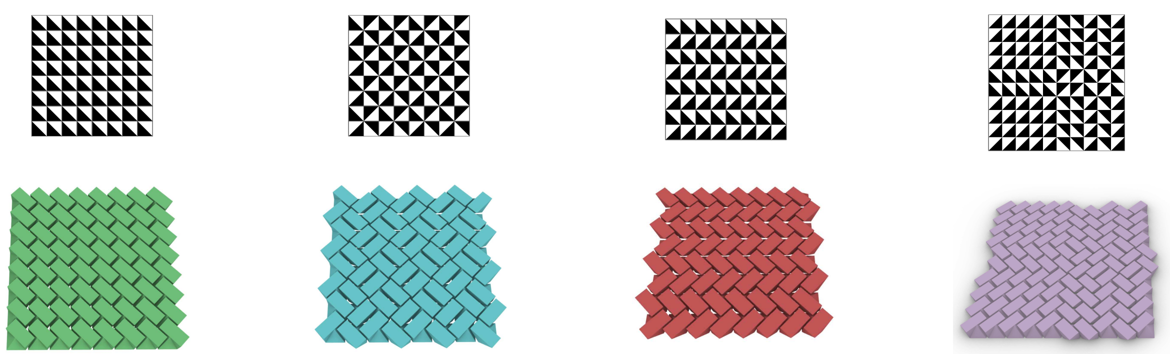


**Figure 4:** *Three periodic assemblies [8] and one aperiodic assembly.*

This approach allows us to estimate the number of possible assemblies with a fixed number of blocks.

**Lemma 1.** *In a planar $m \times n$ grid we have $2^{m+n}$ possible assemblies of the Versatile Block following the assembly rule, i.e. corresponding Truchet tiles only touch at different colours.*

*Proof.* We fill the grid in the following manner: left to right and top to bottom (see Figure 5(a)). This means that in the first box we have four possibilities for the first tile and two possibilities for each box of the first row. For all other rows, only the first tile has two choices, all others are then determined. The number of choices for each box in the grid are given as shown in Figure 5(b). Thus, we have $4 \cdot 2^{n-1} \cdot 2^{m-1} = 2^{m+n}$ possible assemblies. □

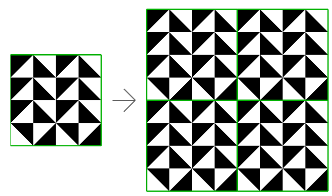
For example, there are over a million possible arrangements in a $10 \times 10$ grid, i.e. $2^{20} = (2^{10})^2 = 1024^2$. It also follows that the set of possible infinite planar assemblies whose boxes are indexed by the countable set $\mathbb{N} \times \mathbb{N}$ is uncountably infinite, since already filling the boxes in the first row can be done in $2^{\mathbb{N}}$ ways. In the proof above we see that copies of each $m \times n$ assembly can themselves be assembled in a periodic fashion by identifying boundary components, see Figure 5(c).

→ ↓
→
→

(a)

| 1 | ⋯ | ⋯ | $n$ | |
|---|---|---|---|---|
| 4 | 2 | ⋯ | 2 | 1 |
| 2 | 1 | ⋯ | 1 | ⋮ |
| ⋮ | ⋮ | ⋱ | ⋮ | ⋮ |
| 2 | 1 | ⋯ | 1 | $m$ |

(b)



(c)

**Figure 5:** *Classifying random planar $m \times n$ assemblies using Truchet Tiles: (a) order of placing tiles, (b) number of possible tiles (c) example of periodic assembly of tiles.*

Apart from planar tilings, the Versatile Block admits different space-tessellations. In Figure 6 we see some of those tessellations with their corresponding translation cells.

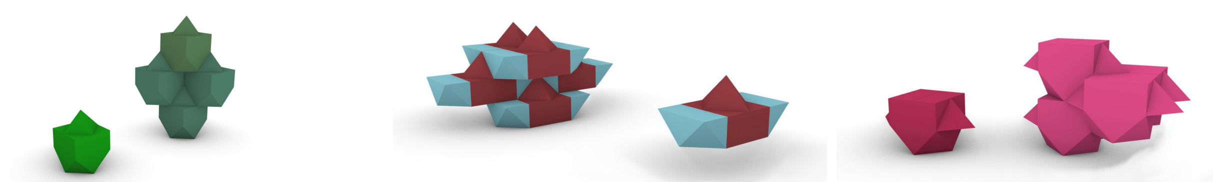


**Figure 6:** *Space-filling assemblies.*

Since, the coordinates of one block lie inside $\mathbb{Z}^3$ it is natural to ask whether coordinates of possible assemblies lie inside the integer lattice.

**Remark 1.** *Let $A$ be an assembly with copies of the Versatile Block such that contact faces are as shown in Figure 2(b). For two blocks, it can be shown that all coordinates lie in $\mathbb{Z}^3$. By induction (removing blocks) from an assembly with $n > 3$ blocks it follows that all coordinates have integer components.*

## Self-Assemblies and Random Assemblies

The concept of self assemblies describes the process in which a set of units forms an organized structure as a consequence of prescribed local interactions and external forces. Inspired by Skylar Tibbits, who is known for his contributions to self assembly and "4D printing" [16], we investigate self assemblies of the Versatile Block by conducting real life and computational experiments. For the study of self assemblies in real life, we produced 3D-printed copies of the Versatile Block with magnets inserted such that the six assemblies given in Figure 2(b) are possible. We then observed how different external forces affected the formation of complex assemblies. For example, Figures 7(a) and 7(b) show the results after introducing controlled trembling to about 50 copies of the Versatile Block under different conditions. The realization of self-assemblies by placing the Versatile Block in environments with different physical constraints such as gravity can serve as a structure generator that can be used as inspiration in civil engineering.

(a) (b) (c)

**Figure 7:** *Self-Assembly experiments: (a) Shaking Box (b) Trembling table (c) Computational.*

For the computational experiments, we employed the computer algebra system GAP [7]. The implementations to study the Versatile Block and its presented properties can be found in the GAP-package SimplicialSurfaces [1]. Using this package we generated random assemblies of copies of the Versatile Block as inspirations for 3D-structures. For each computational experiment we constructed 10000 random assemblies of $n$ copies of the Versatile Block according to Figure 2(b), one example with 100 copies is given in Figure 7(c). By examining their underlying incidence geometry of the vertices, edges and faces, the frequencies of combinatorial non-isomorphic incidence graphs can be counted. For different numbers of blocks, the left table below displays the number of non-isomorphic assemblies i.e. for $n = 4$ there are exactly 2265 assemblies. Table 1 demonstrates that the Versatile Block facilitates a large number of different 3-dimensional assemblies. How vast this number is can be seen when comparing it to the number of non-isomorphic assemblies of copies of a regular tetrahedron (computed with [1]), see Table 1.

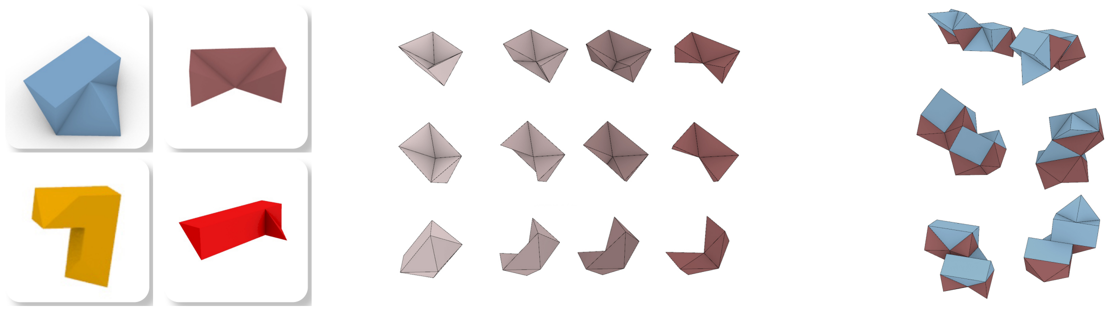
**Table 1:** *Non-isomorphic assemblies of copies of the: Versatile Block (left), tetrahedra (right)*

| $n$ | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|
| | 1 | 6 | 114 | 2265 | 9002 |

| $n$ | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|
| | 1 | 1 | 1 | 3 | 7 |

## Extending the Construction Kit

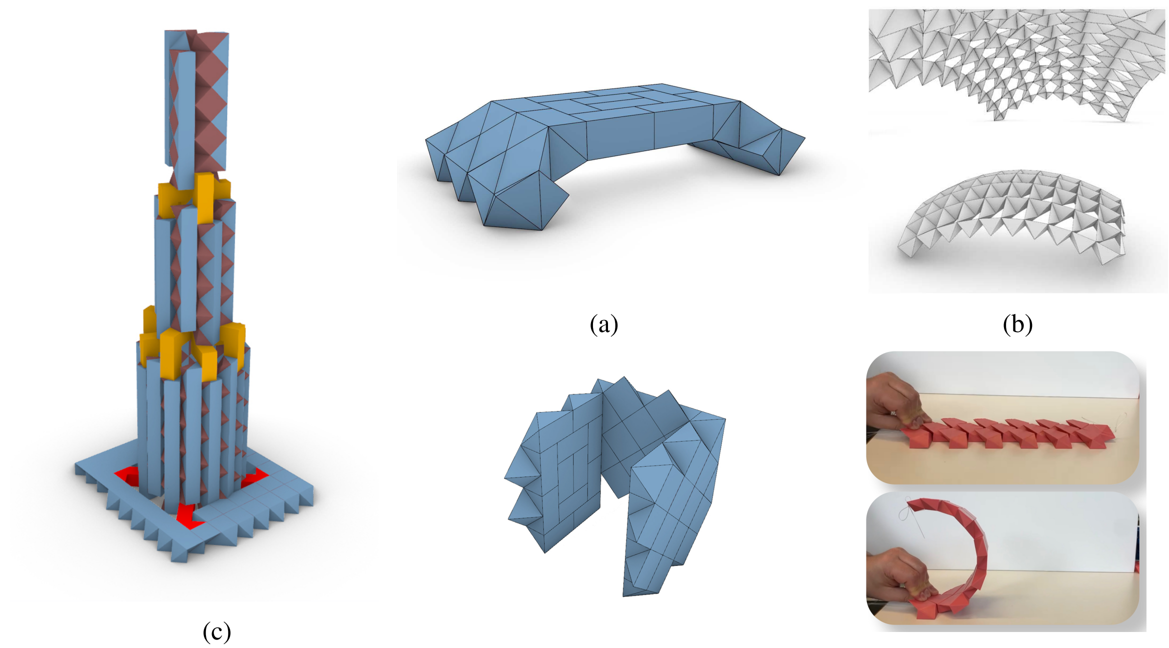
In this section we introduce additional blocks to form a construction-kit to form more general topological interlocking assemblies, i.e. non-planar and curved assemblies. The complete information and functions to generate the blocks can be found in [1]. In order to introduce curvature we can include an angle block for a given angle $\alpha \in [0, 2\pi)$. In Figure 8(b) we present four different versions of an *angle block* for $\alpha = \frac{\pi}{4} = 45°$ and demonstrate how to combine them with the Versatile Block in Figure 8(c), where the Versatile Block is coloured in blue and the angle block is coloured in maroon. The angle blocks allow us to construct arcs and tubular structures, see Figure 9(c). Let $n$ be a natural number with $n \geq 3$. By alternating the Versatile Block with the angle block with angle $\alpha = \frac{2\pi}{n}$ it is possible to construct tubular structures such that the inside forms a regular $2n$-gon. Furthermore, copies of these tubes can be combined to increase the height of this tube. In order to connect tubes with angles of type $\frac{\pi}{4 \cdot m}$ with $m \geq 1$ to planar assemblies, we introduce an additional block (red block in Figure 9(c)). Two tubes can be connected by introducing two additional blocks, called connectors, shown in yellow in Figure 9(c). In total we get the construction-kit presented in Figure 8(a).



(a) (b) (c)

**Figure 8:** *(a) Construction-Kit, (b) Angle blocks for $\alpha = 45°$, (c) Combination of Versatile and angle block.*

## Structural Examples



(a) (b) (c) (d) (e)

**Figure 9:** *Structural Examples: (a) Bridge, (b) Pavilion, (c) Tower, (d) Walls, (e) Arc.*

We identify three possible approaches for architectural and engineering design using the suggested Versatile Block to create TI assemblies: a) When combining only identical copies of the Versatile Block as characterized in the previous chapters, we obtain basic planar building components like facades, walls and floors. Depending on the choice of material, these can function both as load-bearing elements withstanding vertical and horizontal forces (e.g. through wind and earthquakes), as well as non-load bearing partitioning walls or screens. By rotating the block out of the plane we can form basic spatial cells as seen in Figure 6. Following the assembly logic discussed in the previous section (Self-Assembly), complex building envelopes as shown in Figure 7(c) can be approximated. b) By adding a limited set of specific derivations of the original Versatile Block, we can extend the design range of the construction-kit to curved structures like arches, vaults and domes, as shown by introducing the angle block. This allows the design of TI assemblies such as modular bridges (Figure 9(a) frame composed of outer blocks) or tubes, which can be used as arched roof structures. By assembling such tubular structures of different diameters, using different angle blocks, vertical structures such as towers can be designed (Figure 9(c) frame composed of top and bottom blocks and connectors). Since there is a limited set of different blocks, they can be still easily mass-manufactured and assembled. c) The range of possible shapes for TI assemblies can further be extended by parameterizing the Versatile Block and deforming it, to achieve variable curvatures. If the elements are formed individually, this allows for structures to be curved in two directions, see Figure 9(e) and Figure 9(b). However this requires more complex manufacturing and assembly methods, such as 3D-printing with carbon fibre reinforced concrete.

## Realisation and Application to Civil Engineering

In 2022 buildings and construction were responsible for about 38% of greenhouse gas (GHG) emissions [14]. The building sector is also the most resource-intensive, with sand and gravel being the most sought-after resources after water and the most waste producing, responsible for around 34% of global waste [10]. New material solutions like carbon fibre reinforced concrete [2] can lead to both significantly less GHG emissions and more efficient circularity [9]. The suggested TI-block can be applied to construction on three levels: first, it can be viewed as a contemporary extension of traditional masonry structures. It is more durable, since it does not use additional joinery methods, such as mortar, which degrades faster than the bricks themselves [12] and because it reduces crack propagation compared to monolithic components [6]. Currently, the production of bricks has a negative impact on global carbon emissions and deforestation [14]. This can be improved through the use of sustainable materials such as compressed earth blocks or clay and soil cement. These types of circular-use materials can be locally sourced, are low-threshold in production and universally accessible [14]. Due to the reduced strength, they can be used mostly for non-load bearing walls and one-storey buildings. Second, through the use of carbon fibre reinforced concrete and technologies such as concrete 3D-printing, the TI-blocks can be used as structural elements. This facilitates manufacturing modular components, which are hollow, thin-walled and can resist higher tensile and shear forces due to tensile members such as steel or carbon fibre reinforcement. In case of sandwich elements, the forces are transmitted through continuous membranes, therefore lightweight concrete shells can be manufactured [18]. Third, through the use of carbon fibre reinforced polymers, extreme lightweight components and whole buildings can be realised.

## Future Work

In future work the authors will present a package which includes all blocks of the construction-kit and algorithms to create further interlocking blocks for a parametric design approach enabling a digital design workflow for engineers and architects. The angle block in the construction-kit introduces curvature. This leads to the question whether there is a design approach for creating spherical interlocking structures. The authors will deal with this problem in a future publication. The structural examples lead to the question of whether given three-dimensional objects can be discretely approximated using the construction kit.

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