Line-Art Flip Books

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Abstract

I present a flip book exercise wherein participants draw and animate two flip books. The first is an animation of a series of 2D parabolic envelopes, and the second is of a hyperboloid of revolution. In between creating these two flip books, participants should make or observe an existing string art parabolic envelope and a 3D string art hyperboloid. Creating and experimenting with the flip books will enable participants to see how these curved surface can be created with only straight lines.

Introduction

In this paper, I describe a pair of line-art flip book exercises and associated string art. In the first exercise, participants create an animation of a series of parabolic envelopes by connecting dots along two line segments. This exercise helps participants understand how a parabola can be visualized by drawing only straight lines that are tangent to the parabola. In the second exercise, participants create an animation of a projection of a series of hyperboloids by connecting dots along two projected circles. This exercise helps participants visualize how the three-dimensional hyperboloid of revolution can be projected onto 2D space. Participants should also discover that despite its apparent curvature, the hyperboloid is a ruled surface and can be created out of only straight lines. The exercise forces participants to spend more time contemplating the relationship between the hyperboloid, a cylinder, and two cones than they are likely to if presented with a fixed model. Two string art activities to generate string art 2D parabolic envelopes and 3D hyperboloids are described and recommended as an interlude between the two flip book exercises.

After completing the proposed workshop, participants should understand how to use line art to create geometric curves and how to manipulate the position of their lines to create different curves.

This paper extends upon our 2013 and 2016 workshops on polyhedral and fractal flip books respectively [6, 7]. Here, just as in the previous workshops, the process of developing and flipping flip books helps participants gain mathematical intuition. However, the methodology of flip book creation is somewhat different to accommodate the different material being covered.

Related Work

Animations are a valuable and commonly used tool for illustrating all kinds of mathematical concepts. They are especially useful for visualizing three dimensional objects effectively on a screen. However, most modern mathematical animations are presented on screens and generated by computer.

Animated flip books were first patented in 1868 [9] and have primarily been used for entertainment. Recently, several authors have advanced approaches to using flip books for math and science instruction [3, 10]. To my knowledge, however, our series of flip book workshops are the first to use animated flip books for teaching geometric concepts [6, 7].

Hand drawn line art and similar string art creations are commonly seen in school art classes [1, 2, 8, 11, 14]. Drawing parabolas in a variety of ways is part of the standard math curriculum. There is a delightful method for creating a parabolic envelope using origami by defining a dot as the focus of the parabola and using one edge of the paper as the directrix and repeatedly folding different points along the edge of the paper to the dot [2]. Hyperboloids are also easy to generate and commonly seen in math art, generally in 3D form rather than projected [4]. However, I believe that hand-drawn 2D flip book animation of these concepts is novel.

Workshop Description

Materials

Participants are provided with uncut template sheets that should be drawn on following the template and then cut and stapled to assemble the final flip books. These template sheets can be found on my website [5]. Each person should have one copy of the parabola template (two pages) and two to three copies of the hyperboloid template for this workshop. This differs from our previous flip book workshops where participants were provided with pre-assembled flip books with a template provided only on the final page [6, 7]. Pre-assembled flip books have the advantage that one can draw on top of the previous sheet to ensure alignment of the drawings from page to page. This workshop requires extensive straight-edge use to create accurate drawings and it is substantially easier to use the straight edge on a flat sheet than an assembled book. To ensure that the drawings align once the flip books are assembled, I provide template lines for every page of these flip books.

While any writing or drawing implement will work, thin colorful pens or pencils are preferred for creating dynamic, multi-colored flip books with distinct lines. Participants should also have or be provided with pencils, erasers, and straight edges.

For flip book assembly, participants need access to a pair of scissors and/or a paper cutter and a stapler.

For the optional string art activities, participants will need colorful string or yarn, stiff cardboard, very stiff paper plates or similar, and scissors. They will also want something to hold their plates apart while the hyperboloid model is being assembled. We used empty paper towel tubes.

Parabolic Envelopes Flip Book

The first flip book that participants should work on is the parabola flip book. The template for this flip book involves dots along two connected line segments. One line segment rotates 180 degrees in 15 degree increments relative to the other line segment over the course of the flip book, such that the segments begin colinear and adjacent and end on top of each other.

Participants can lightly number the dots along each line with a pencil. They should start at one end of the template and number the dots sequentially 1 to 12, and then 1 to 12 again as in Figure 1. Using a straight-edge, participants should draw lines connecting pairs of dots with the same numbers. Some people may find it sufficient to mark only the first dot on each line and then to draw subsequent lines sequentially from there.

It is up to the individual whether they want to draw these lines merely between the marked dots as in Figure 2(a) or if they wish to extend past the dots to fill as much of each flip book page as desired as in Figure 2(b).

Once all of the lines have been drawn, the penciled markings can be erased and it is time to cut out and assemble the flip books.

Flip Book Assembly

Each sheet of the template should be cut into eight flip book pages along the indicated lines. The sheets should be arranged in order such that the left edges (these should be the edges with more white space on the template) are aligned. This left edge can now be stapled together to create a flip book as in Figure 3.

To aid in flipping, it is helpful, but not necessary, to make a bias-cut along the edge opposite the binding.

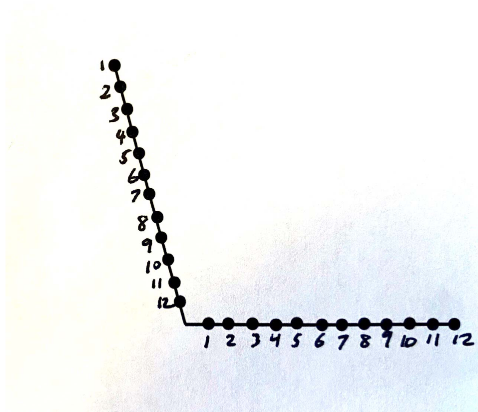
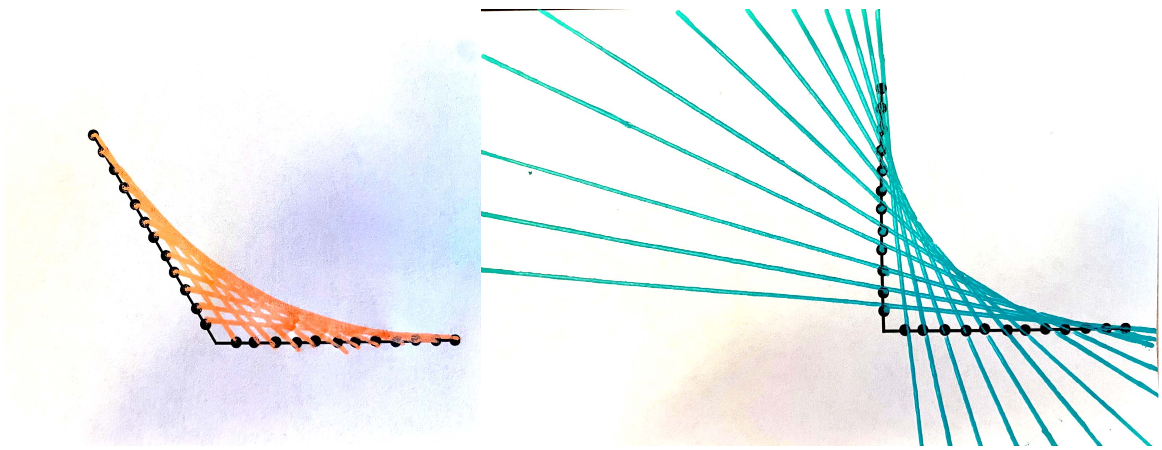


Figure 1: *A parabola template annotated with numbers to aid in correct line drawing.*



(a) **(b)**

Figure 2: *It is up to the participant whether they want to end the lines forming their parabolic envelopes at the dots as on the left or extend them to the edges of the paper as on the right.*

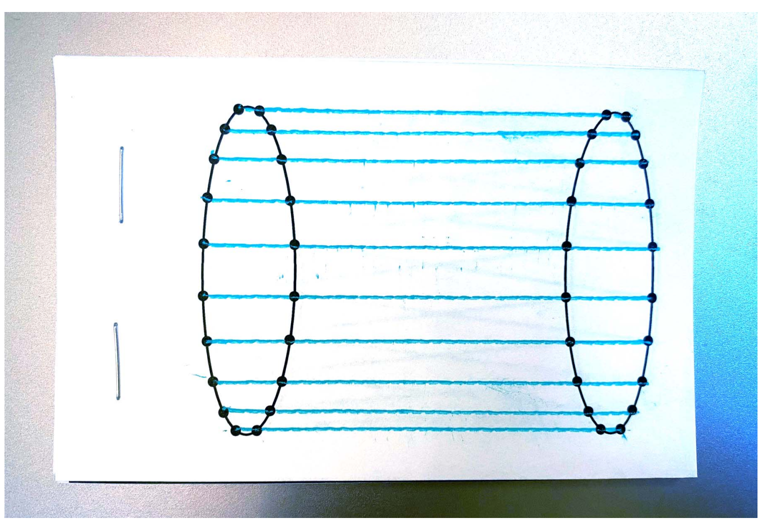


Figure 3: *A completed and assembled flip book*

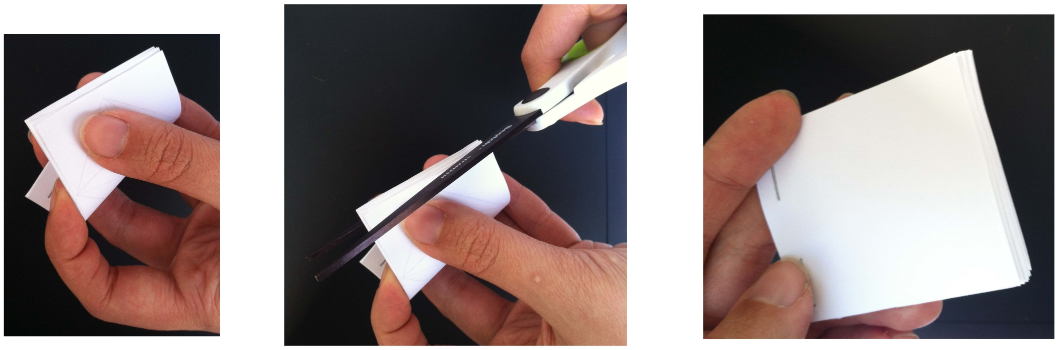


Figure 4: *Demonstration of bias-cutting the flipping edge*

Figure 5: *The front and back of a cardboard and string parabola*

To make a bias-cut, bend the flip book into a U-shape, as shown in Figure 4. Then, make a straight cut. This will result in a bias-cut flipping edge. Note that a bias-cut flipping edge will flip more easily than a straight-cut flipping edge, but will flip in only one direction.

String Art Activities

Before starting the hyperboloid flip book, it is very helpful to first see a 3D model of a hyperboloid. It is straightforward to generate string art models of both the parabola and the hyperboloid, and I recommend that string art either be part of the class or that the instructor spend a little bit of time generating the string art models described below at home to show to the class.

For the parabola model, start with a square piece of stiff cardboard and cut evenly spaced slits along two sides. String can then be woven through the slits to generate a parabola. A single length of string can be used by weaving to the next slit on the backside of the cardboard. The front and back of an assembled string art parabolic envelope can be seen in Figure 5.

For the hyperboloid model, start with two circular pieces of stiff cardboard or stiff round paper plates. Cut each plate with a matching set of evenly spaced slits as in Figure 6(a). Using a paper towel tube or similar to keep the circles separated, weave the string between matching slits in the circles to generate a cylinder model as in Figure 6(b). The tube can now be removed and by holding the two circles apart and twisting

(a) (b)

Figure 6: *Construction and assembly photos for the string hyperboloid*

them relative to each other it is possible to generate a series of three dimensional hyperboloids as in Figure 7.

Hyperboloid of Revolution Flip Book

The second flip book that participants should work on is the hyperboloid flip book. The template for this flip book involves dots along two ellipses. These ellipses are projections of circles at the top and bottom of the hyperboloid. While each page on this flip book looks identical in the template, we can think of the activity as involving two circles that rotate with respect to each other such that each line drawn constantly connects the same two dots.

As in the parabola flip book, it is helpful to lightly number the right and left circles in pencil before drawing connecting lines in pen. These pencil marks can be erased once all the lines are drawn. I recommend numbering the right circle consistently across all the pages and numbering the left circle the same as the right circle in the first image and then rotating all the numbers by one dot in each subsequent image. Participants may also want to lightly number each page to make sure that the flip books are assembled in order. Again, some people may find it sufficient to merely annotate the first set of matching dots on each picture and then draw lines sequentially from there.

At this point, the exercise is identical to the previous flip book and involves simply using a straightedge to connect like numbered dots on each page.

As there are 20 dots around each circle, there are 20 possible rotations that can be generated in this exercise, however it is not necessary to complete all 20 pages to generate a compelling flip book. After completing 10 pages the flip book should already animate from a cylinder to two cones traversing through a series of hyperboloids. Interestingly, unlike in the real physical 3D string art model described earlier, these lines can pass through each other, so after reaching the double cone state seen in Figure 9(a) the drawn flip books can continue to rotate back to the cylindrical state.

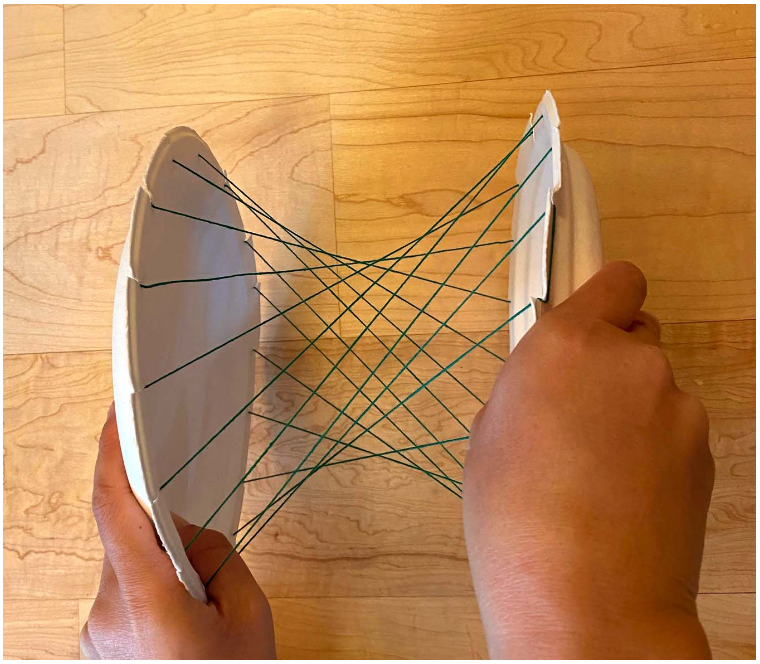


Figure 7: *Completed and twisted plate hyperboloid*

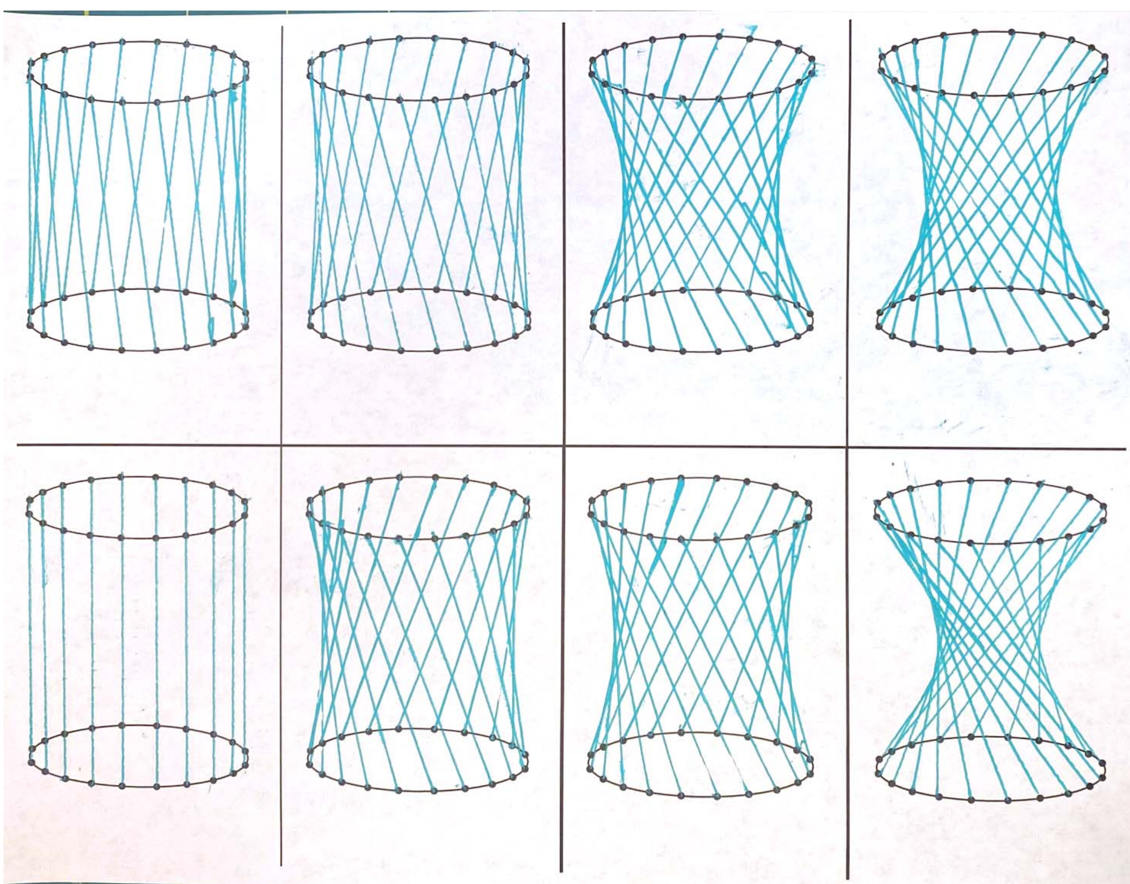
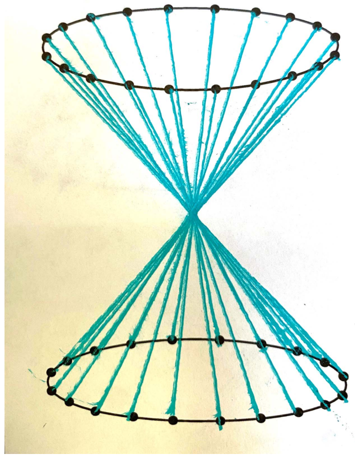
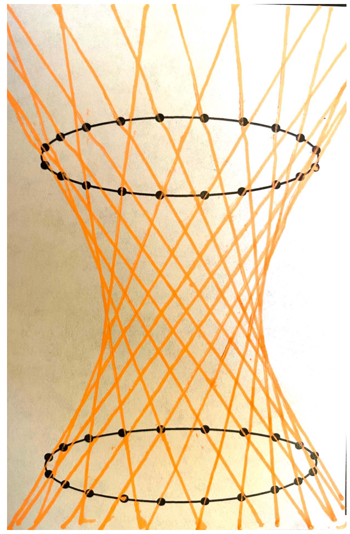


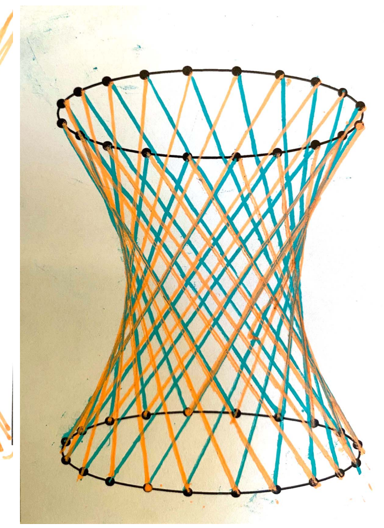
Figure 8: *A completed template page for the hyperboloid flip book, ready to be cut and assembled.*



(a) *One of the two limit cases of the hyperboloid is two cones.*



(b) *The lines defining the hyperboloid have been extended to the end of the flip book page.*



(c) *A two colored hyperboloid ruled in both directions.*

Figure 9: *Three potential flip book pages.*

As in the hyperbolic envelope exercise, even with the provided templates there is an opportunity for creativity. For example, participants can choose to continue their lines past the provided markings as in Figure 9(b) or choose to also draw the opposite direction rotation in a different color on top of their initial rotation to show that the hyperboloid is doubly ruled as in Figure 9(c).

On completion of the drawing stage of this flip book, refer to the assembly directions above to assemble the second flip book.

Discussion

This set of flip book and string art activities allow a lot of freedom to approach the same activity from a more mathematical, more technical, or more artistic point of view depending on the audience. While this workshop can be presented in its entirety in one session, it would also make sense to split it up into multiple classes so as to allow time to discuss key points for each activity.

Much has already been written about how to frame an activity around line-art hyperbolic envelopes. As a math activity for younger audiences, 3Doodler provides an example of how to focus the parabolic envelope part of the workshop [1]. One can also easily find more mathematical and technical parabolic envelope exercises [12] or more art-focused ones [13] that can be used as a basis to frame the hyperbolic envelope flip book for different audiences.

Less has been written about hyperboloid activities, and that which has been written seems to be focused on variations upon creating a hyperboloid from sticks as described by George Hart and Elizabeth Heathfield [4].

Participants should take away from the parabola part of this workshop that it is possible to create a curve using only straight lines. The curve we are generating approximates a parabola, the lines that surround and define the parabola are known as a *parabolic envelope* because they "envelope" the actual parabolic curve. In a parabolic envelope, each line is *tangent* to the parabola, ie. it touches the parabola at exactly one point.

From the string art part of the workshop, participants should observe that in two dimensions, obviously a straight line cannot be part of a curve, however, in three dimensions each line is actually on the hyperboloid. A surface that can be composed entirely of straight lines is known as *ruled*. The hyperboloid is actually *doubly ruled*, for each point on the hyperboloid there are actually two different intersecting straight lines that touch it and are on the hyperboloid. This hyperboloid is properly known as a hyperboloid of revolution, as any one of the strings could be revolved around the center of the hyperboloid to align with every other string and generate this shape.

From the hyperboloid flip book part of the workshop, participants should notice that the hyperboloid is a three dimensional object, but, by making a flip book of it, we are projecting it onto two dimensional space. As a result, the template circles are actually ellipses as a result of this projection. A straight down projection would make the circles actually circular, but would make it very difficult to tell that the object being drawn was a hyperboloid. Further discussion on how a projection differs from the real object would be valuable.

For both of the flip books, participants should observe that the limit cases of the drawings are not parabolas or hyperboloids, but rather lines, cylinders, and cones. In other words, there is a natural animation between a cylinder and two cones that progresses through a series of hyperboloids.

I encourage participants to use what they have learned from this workshop to make new and different line-art flip books and string art.

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