Sculpting Mapping Cylinders: Seamless Crochet of Topological Surfaces

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Abstract

In this workshop we will learn how to crochet topologically nontrivial surfaces by building appropriate ribbon graphs from foundation chains. This technique perfectly matches the concept of a mapping cylinder in algebraic topology. Examples include surfaces whose boundaries are torus knots/links and regular polylinks.

Introduction

Many crafts–including crochet, knitting, beading, and the coiling method in weaving and ceramics–involve building rounds or rows on top of previous ones. Once the foundation is laid, such an act is simply making the surface larger without changing the topology. One might ask what actions in these crafts decide the topology.

Consider a simple paper project of gluing the Möbius band. One first cuts a long strip and then joins the ends together after a 180-degree twist. For materials other than paper, in theory such a project can be finished in a similar fashion, provided there are ways of twisting and joining of the material, although this is by no means the only approach. Fortunately for us, crochet is among such crafts where both actions are possible. In particular, one can consider a foundation chain to be a ribbon and two separate chains or different parts of the same chain can join by identifying chain stitches. This flexibility makes crochet an ideal way to make topologically interesting surfaces.

Daina Taimina [10] and Gabriele Meyer [7] have constructed stunning crocheted hyperbolic plane sculptures. Hinke M. Osinga and Bernd Krauskopf [8] have turned chaotic system equations into fascinating a real-life object called a Lorenz manifold. In this workshop we will extend crochet further into the topological realm. The technique that we employ can be described as the physical implementation of a mapping cylinder, which we will define later. Figure 1 and [3] show a few examples. This work is a continuation of inspiring topological works by Matthew Wright [11], Hanne Kekkonen [6] and Moira Chas [2].

(a) (b) (c) (d)

Figure 1: *Surfaces bounded by: (a) the (4,3)-torus knot, (b) two coaxial torus knots, (c) a link of 6 components with chiral octahedral symmetry, and (d) the regular polylink of 10 triangles.*

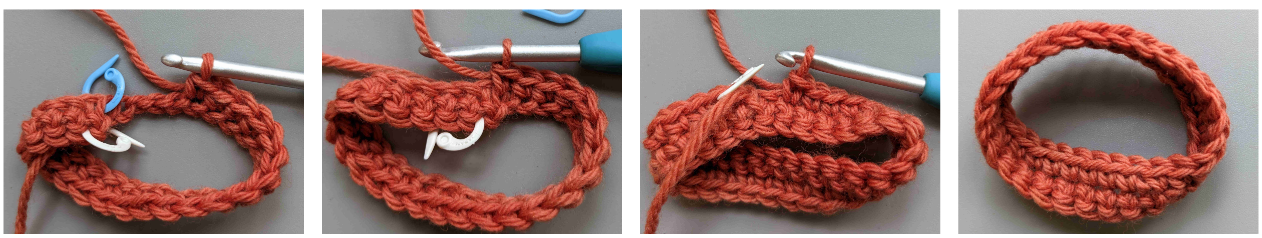
Warming up with the Möbius Band

Throughout this paper, a *twist* always means 180 degrees unless otherwise noted. We will use US crochet terminology.

Materials needed for the projects in this paper include two colors of worsted-weight yarn (one for the body and one for the boundary) and 30 inches of 20 or 22 gauge half-hard copper wire. We will use a size 7 (4.5mm) crochet hook and a few split ring stitch markers. Scissors, darning needles, wire cutters, and pliers can be shared among participants. A blank temari ball or a scrap yarn ball and two blocking pins are helpful but they are not essential.

We will start by making a Möbius band. Step-by-step instructions for this project are shown in Figure 2. We need one foundation chain of about 27 stitches (Figure 2(a)). Twisting it and joining the end chain stitches produces a narrow Möbius band (Figure 2(b)). We then add a round of single crochet stitches (Figures 2(c)-2(g)) to make it prettier and more stable. The final work shown in Figure 2(h) produces a satisfying result.

(a) *Make a foundation chain of about 27 stitches.* (b) *Twist it and join the end chains together by a blue stitch marker.* (c) *Insert crochet hook into the joined stitch and chain one.* (d) *Start single crocheting and mark the first single stitch.*



(e) *Encounter joined chain stitch with the blue stitch marker.* (f) *Crochet into the joined chain stitch and remove the marker.* (g) *Stop crocheting before the marked first single stitch.* (h) *The final work.*

Figure 2: *Making a Möbius band.*

It is important to remember that the two end chain stitches that were joined should always be treated as if they were a single stitch whenever we crochet into them. This applies to all joined stitches in all projects discussed in this paper. The use of stitch markers is extremely important to hold the joined stitches together before we crochet into them.

One might notice the constant need to turn our work when crocheting the only round. This is a characteristic of this style of topological crochet: the natural flow of the chains guides our crochet hooks. Care has to be taken when crocheting so that such flow is respected and no unwanted local twists are created. This can be done by always holding the piece so that

- you have an unobstructed front view of the segment of the row you are working on and
- the working yarn is in the back of that row.

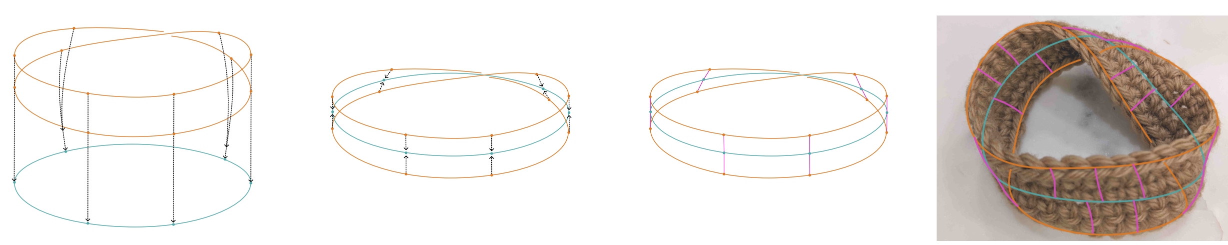
One goal of this project is to get ourselves familiarized with the actions of chain twist and stitch join. These are key ingredients for building more complicated models.

This exercise provides a simple example of a non-trivial mapping cylinder, the precise definition of which can be found in standard algebraic topology textbooks, such as [5]. Here we are going to explain it in plain words associated with crafts. For a given shape X, we can construct a *cylinder* as $X \times I$, where I is topologically a line segment. In the craft world, the direction in which we build rounds and the work grows is that of I. For example, in knitting and crochet when we make a simple round cowl, X is a circle S^1, and in that case $X \times I$ is simply the whole cowl, which is an *annulus* in topology. Also, imagine in ceramics laying down the first coil with the shape ∞, and building rounds of coils with the same shape on top, we would get a sculpture that is a topological cylinder $\infty \times I$, which looks like two annuli glued along a line. Vaguely speaking, a topological cylinder $X \times I$ is a shape with a growth direction, so that at each given point in the direction, it is topologically X. Intuitively, a cylinder $X \times I$ is simply is a widened version of X.

To make it more interesting, we need to add more ingredients. Let's say there is another shape Y and a map f from X to Y. A *mapping cylinder* for such a map is the cylinder $X \times I$ with its bottom X replaced by Y. The way the body of the cylinder continuously connects to the bottom Y is decided by f. Making the projects in this workshop will help us understand these topological objects.

The way we crocheted the Möbius band, the only round we made (see Figures 2(c)-2(g)) is a circle. Therefore using the notation above, X is a circle and $X \times I$ is an annulus. But a Möbius band is not such a shape. The key lies in Y and how X maps to Y. To see Y, we need to reverse the operation of construction and trace back to the beginning of I. In this case it means we need to imagine removing rounds and rounds, till our work is reduced to the thinnest possible. The result is a circle that lies in the center of the band.

Even though X and Y are both circles, it doesn't imply the resulting mapping cylinder would simply be $X \times I$. We still need to work out the map from X to Y. When crocheting the only round of X (Figures 2(c)-2(g)), we circled along the center Y twice, as if X wraps along Y twice. Such a multiple wrapping is called a *degree-n covering*, where n is the number of wraps. Figures 3(a) and 3(b) show abstract images of such a mapping. When we approach the bottom of $X \times I$, the two points in X that map to the same point in Y will be identified as one point. Figure 3(c) illustrates how the Möbius band results after such identification. The participants can deepen their understanding by cutting a paper Möbius band along its center circle and observing that a new circle of double size emerges. Finally, Figure 3(d) demonstrates how the ingredients in the definition of a mapping cylinder match with our stitch work. In particular, Y is the center of the foundation chain after joining the end stitches, X is each round, and I lines up with the posts of the stitches.



(a) *A degree-2 covering map from the orange circle to the blue circle.*

(b) *An alternative way of drawing the same map.*

(c) *Replacing all abstract arrows by I makes a mapping cylinder.*

(d) *Match between sketch and a crocheted model.*

Figure 3: *Understanding the Möbius band as a mapping cylinder.*

Hence, a Möbius band is topologically the mapping cylinder of a degree-2 covering from a circle to another circle. This abstraction might be a little overwhelming to some, but with it also comes the rewarding power of generalization, which will help us understand and design a countless number of sculptures.

The Trefoil Knot

Now that we are more or less familiar with the twist and join moves, let's look at the main project in this workshop. We will make a surface bounded by the elegant trefoil knot (see Figure 5(a)). We will crochet three rounds on top of the foundation chain. The third step uses treble crochet stitches (tr) as well as adding and combining of stitches. Those who are newer to crochet are free to skip this step during their first attempt.

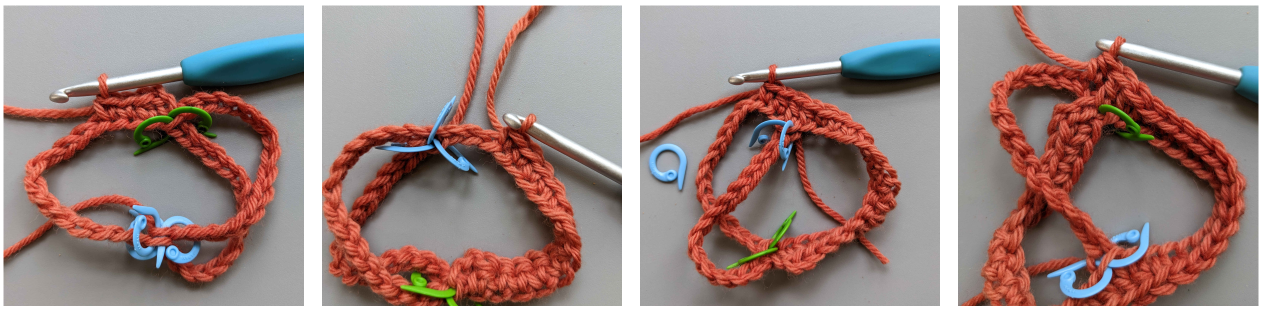
- **Build the foundation.** Chain 49 stitches plus a few extra that we will remove later. Remove the crochet hook. Starting from the first stitch, mark every 16th chain stitch, four in total (at numbers 1, 17, 33 and 49). If necessary, secure the first chain stitch in place on a temari ball or a yarn ball. Twist once to secure the second marked chain stitch (Figure 4(a)). Twist again to join the first chain stitch with the third marked chain stitch. Without a temari ball this can be done by joining the first and third marked stitches with a 360-degree twist. Place three stitch markers at this join (Figure 4(b)). Then twist once more to join the second marked chain stitch with the fourth marked chain stitch. Place two stitch markers at this join (Figure 4(c)). We have built a graph with two vertices, or *branch points* as we will call them, with three edges connecting them. Make sure all the edges twist the same direction.
- **Round 1: remove the stitch markers.** Undo all the extra chain stitches, insert crochet hook into the second join and chain 2 (ch2) to get ready for round 1 (Figure 4(d)). Crochet round 1 with the following guideline: double crochet (dc) at either branch point, half-double crochet (hdc) at the two neighbors of either branch point and single crochet (sc) the rest (Figures 4(e)-4(i)). In total we will go through six sides of the edges, each with a half-double crochet stitches at each end, and 13 single crochet stitches in between. There will also be three double crochet stitches around each branch point. Remove a stitch marker whenever we crochet past one. Use a slip stitch to end the round. For those who are newer to crochet, dc and hdc may be replaced by sc all around.
- **Round 2: beautify.** This step is optional. Place the work so that you can trace the trefoil knot. For any of the three twisted edges, there is one side facing outward and one inward. For the outward side we will crochet taller and more stitches and for the inward side the opposite. Double check there are 16 total stitches on each side of the edges. We are going to crochet in a symmetric way. On the outer side, for the eight stitches from the branch point to the center, crochet as follows: dc for the first four, 2dc for the next two, 2tr for the next two – I'll call this an outer half sequence (ohs). On the inner side, for the eight stitches from the branch point to the center, crochet: dc, hdc, sc, sc, sc2tog, sc2tog – I'll call this an inner half sequence (ihs). Starting from any branch point, we'll crochet the concatenated sequence of ohs (Figure 4(k)), reversed ohs (Figure 4(l)), ihs and reversed ihs three times (Figure 4(m)). Fasten off. This round is where the whole project resembles ordinary crochet the most – we choose the stitch arrangement to vary the size or to add other decorative aspects. The participants are encouraged to experiment with different arrangements to achieve different visual effects.
- **Round 3: color and wire the boundary.** This is the most exciting step as our sculpture will gradually reveal its three dimensional look as we work along its boundary. Cut the wire to the correct length. Use the second color to single crochet around the wire and along the boundary, starting from a point that is close to the beginning of an ohs (Figure 4(n)). This point is chosen so the wire ends will hide better. Stitches can be added when the boundary curvature is high, as long as the choice is symmetric. I choose to add three on each outer side. Before finishing the last inner side, join the wire ends and trim the extra (Figure 4(o)). After finishing, adjust the boundary so that it looks smooth and symmetric (Figure 4(p)). If time is running short during the workshop, one can skip the wiring and make the boundary with the colored yarn only. Safety alert: be very careful of the sharp wire ends. It's always safer to join the wire ends before crocheting but it can be tricky to do. If you decide to crochet with loose wire ends, you can bend about one cm of wire back to make the ends not so dangerous.

(a) *Two branch points with one twisted edge between them.*

(b) *Three stitch markers around the first branch point.*

(c) *Three twisted edges connecting two branch points.*

(d) *Remove extra chains, insert the crochet hook back and chain 2 for dc.*



(e) *Start crocheting with a hdc followed by single crochet stitches*

(f) *Approaching the first branch point.*

(g) *Dc at the branch point hdc at both sides of it.*

(h) *Encounter the second branch point and repeat.*

(i) *Back at the first branch point. Only one stitch marker is left.*

(j) *Starting round 2 with chain 2 for dc.*

(k) *After the first ohs.*

(l) *A reversed ohs completes the first side.*

(m) *The whole round 2 finished.*

(n) *Crochet around the wire with the second color.*

(o) *Join the wire ends on the last inner side.*

(p) *The final work.*

Figure 4: *Making a surface bounded by the trefoil knot.*

To summarize, in the first two steps we build the foundation chain graph and reinforce it. In the third step (Round 2) we add artistic elements, and in the fourth step (Round 3) we finish the work by decorating it with desired colors and using metal to support the whole surface. The first, second, and fourth steps are essential, while the third step can be omitted completely.

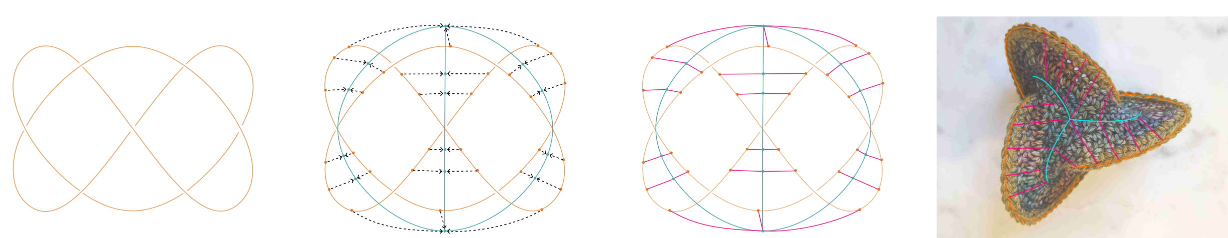
I found the following tips are very helpful when learning this style of crochet.

- Extensive stitch marking is always necessary, even when we think we fully understand. Since there is no natural global coordinate system it is very easy to forget how the edges around each branch point should line up. It is worthwhile to use a stitch marker between every pair of neighboring edges.
- To reduce the confusion of stitch counting, it's better to crochet in rounds instead of a spiral.
- Let the wire bend along the boundary without too much intrinsic twist, because when the wire has high twist resistance it'll change the bend and introduce unnecessary tension on our stitches.
- Since the boundary is usually a non-planar curve, when we crochet along it we need to reposition the yarn ball back and forth; a smaller yarn ball is easier to maneuver. Sometimes the stitches are hard to reach; a long Tunisian crochet hook can help. Be extra careful not to miss any stitches.
- Make sure cats and toddlers are otherwise entertained.

As promised, this surface is another example of a mapping cylinder. The round we make on top of the foundation chains is a trefoil knot, therefore X is topologically a circle. Y is recovered by shrinking the whole work to zero width, and in this case it's a graph G consisting of two vertices with three edges in between. As for the map from X to Y, notice that like before, each edge gets crocheted into twice, once on each side. However, each branch point gets crocheted into three times. Therefore the map from X to Y is almost a degree-2 covering except at the two vertices of Y (see Figures 5(b) and 5(c)). Mathematically we call such a map a *branched covering of degree-2*. The word *branch* is especially apt, as around the branch points the edges stretch out just like tree branches.

In conclusion, the sculpture we made is the mapping cylinder of a branched covering of degree-2 from a circle to a graph with two vertices and three edges. Figure 5(d) shows how the concept lines up with our crochet stitches around one of the branch points.

To have X, Y, and the branched degree-2 covering map between them in our mind is beneficial for the successful execution of our project. In particular, during the initial setup of the foundation chains, we form a graph that we can easily think of as just the graph Y, since the chains are narrow. Instead, the graph is made of bands joined at certain tiny topological disks, which makes it mathematically a *ribbon graph*. It contains Y in its center, X at its boundary, and is a mapping cylinder. To reduce the possible confusion when using the word graph, let's call Y the *skeletal graph* of the ribbon graph. In a ribbon graph, the edges need to respect their cyclic order around each disk they join. Diligent stitch marking helps prevent potential mistakes.



(a) *A right-handed trefoil knot.*

(b) *A branched covering of degree-2 from the trefoil knot to graph G.*

(c) *Replacing each abstract arrow by I makes a mapping cylinder.*

(d) *The top view of the match between sketch and a real model.*

Figure 5: *Understanding our sculpture as a mapping cylinder.*

Examples Based on Planar Graphs

A knot or link diagram traces edges of a projection of a surface onto the plane [1]. If we consider the unbounded region in the plane as a bounded one in S^2 then there are two choices for such surface. Each surface corresponds to a ribbon graph, which when all edges are untwisted, is planar. A simple planar ribbon graph is homeomorphic to a polyhedron with a puncture on each face. The two surface choices for each knot/link diagram correspond to dual polyhedra. Hence, when searching for a good example to crochet, we can start from a polyhedron. Imagine puncturing a hole in each face and deforming it into a ribbon graph. Then twist the edges however you want to build a foundation chain graph. The resulting crocheted surface will have a knot or link as its boundary and the 1-skeleton of the polyhedron as its skeletal graph. Dual polyhedra with matching twists give rise to different surfaces with the same boundary. Table 1 is a list of examples the participants are encouraged to make at home. Figure 6 shows some resulting surfaces. Between two dual polyhdera, the one with lower valences is easier to work with. In [9] Carlo H. Séquin analyzed Charles O. Perry's work, all of which are based on planar skeletal graphs.

Table 1: *Examples of polyhedron-based models.*

Polyhedra	Twists	Boundaries
adjoining one n-gonal pyramid and m of n-gonal prism(s)	opposite directions for vertical and horizontal edges	$(3+2m, \pm n)$-torus knot/link
adjoining m of n-gonal prism(s)	opposite directions for vertical and horizontal edges	$(2+2m, \pm n)$-torus knot/link
dodecahedron/icosahedron	the same twist for every edge	6-pentagon link as in [4]
cube/octahedron	the same twist for every edge	4-triangle link also in [4]
tetrahedron	the same twist for every edge	Borromean rings

(a) (b) (c) (d)

Figure 6: *Surfaces bounded by: (a) the (3,-3)-torus link, (b) the (3,-4)-torus knot, (c) the Borromean rings, and (d) the regular polylink of 6 pentagons.*

Once a ribbon graph is chosen, different choices of twists result in very different surfaces. For example, when we crochet the trefoil knot, if we turn two edges 360 degrees and the third edge 180 degrees in the opposite direction, we get a piece that is topologically equivalent to the punctured Sudanese Möbius band.

The mapping cylinders we have described shrink to their skeletal graphs, which are one dimensional, therefore S^2 or a genus-n torus are not among them. The only thing missing in our method are some 2-cells, or caps. Suppose we have already made a surface with a certain link on the boundary; as long as there is a component that can shrink to a point without obstruction, we can continue crocheting rounds along that circle. Gradually, we shrink the additional rounds to a tiny circle that we can fasten off. This modification results in a structure called a *double mapping cylinder*. Therefore with our mapping cylinder technique and a slight modification, arbitrary topological surfaces can be created. The participants are encouraged to experiment with more surfaces, such as a Sudanese Möbius band, by crocheting a cap on the punctured version.

Conclusions

The terminology we used in these projects are concepts in algebraic topology. We've made exciting real-life examples of otherwise abstract and less-well-known theoretical constructions. The beauty of this technique is that it builds a perfect and smooth connection between crochet and algebraic topology, and the resulting work can be quite artistic. In addition to topology, in these projects one can learn about symmetry, polyhedra, minimal surfaces, graphs, and ribbons. They are extremely mathematically rewarding.

As mentioned at the end of the last section, there is no limit to what we can make. Any surface can be crocheted by carefully extracting its ribbon graph structure. Or one can start from any interesting ribbon graph and see what surprise awaits us. In my own exploration, I have enjoyed working both ways.

With proper tension from the wire on the boundary and a nice arrangement of stitches inside, the surface will stretch naturally and resemble a minimal surface elegantly. When that state is achieved, the wire and stitches maintain a harmonious balance without outcompeting each other, giving us a hint of the beauty hidden behind the veils of this world. Perhaps one day, topological sculptures will be common projects in knitting circles, where people exchange passionate insights about knots, links and mapping cylinders.

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