Connections between Hitomezashi Patterns and Truchet Tiling

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Abstract

Many visual patterns are often created on an underlying grid of squares. Truchet tiles are squares decorated with two quarter-circle arcs that can be used to create fields of meandering patterns. Hitomezashi is a type of traditional Japanese sashiko embroidery, often white thread on blue fabric, where large running stitches follow orthogonal grid lines to create regular patterns. Hitomezashi forms two complementary patterns, one on each side of the fabric. Interestingly, one can overlay Truchet curves on a hitomezashi pattern such that half of the Truchet arcs follow the hitomezashi pattern and the other half of the Truchet arcs follow the obverse hitomezashi pattern. However, an arbitrary Truchet pattern will not always have a corresponding hitomezashi pattern. Examples of each pattern type and relationships are shown and discussed.

Introduction

In 1704, Father Sébastien Truchet investigated the patterns obtainable from a single square tile that was bisected along a diagonal into two isosceles right triangles, one black and the other white [5]. This tile has four possible orientations. Truchet enumerated all possible patterns arising from pairs of tiles, reducing the 64 possible combinations to 10 unique equivalence classes after rotations. He also illustrated several intricate patterns made from larger blocks of tiles.

In 1987, C.S. Smith provided an updated commentary on Truchet's work along with new ideas including the use of a random tiling rather than a structured pattern [3]. Smith also included a variant of the Truchet tile that replaced the triangular segmentation with two quarter-circle arcs, resulting in a tiling that is comprised of an aesthetically pleasing, meandering set of continuous and differentiable curves. While not created or studied by Truchet, this arced variant is commonly known as a Truchet tile. Smith does not cite previous work for the arced Truchet tile concept, but does refer to a 1977 Martin Gardner column which describes similar arcs on aperiodic Penrose tiles [1]. Example random Truchet tilings are shown in Figure 1.

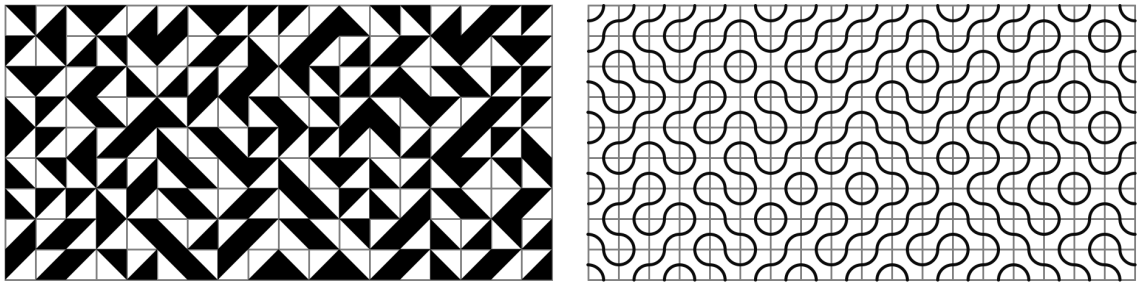


Figure 1: *Example random tilings: Truchet's original tiles (left) and the Smith variant (right).*

Sashiko is a Japanese decorative mending or upcycling technique that uses many small stitches to combine two or three layers of fabric; it dates to the Edo era (1615–1868) [2]. There are many different styles

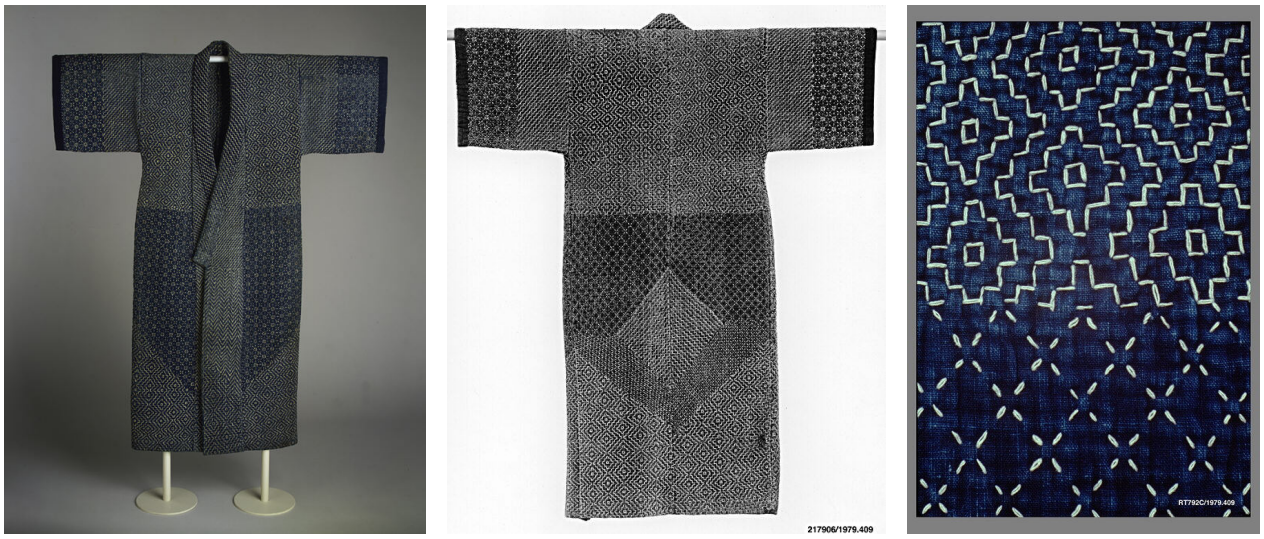


Figure 2: *An example sashiko kimono (mid-19th century) with white embroidery on indigo fabric [4].*

that evoke natural patterns such as flowers and falling snowflakes. While many patterns are very geometric, some are more flowing in their style. An example is shown in Figure 2.

Hitomezashi is a type of traditional Japanese sashiko embroidery, often using white thread on blue or indigo fabric, where large running stitches follow orthogonal grid lines to create regular patterns [2]. Horizontal and vertical unit-length running stitches are placed along a uniform square grid, alternating between the top and back of the fabric. This process is depicted in Figure 3. Different patterns are created by varying the thread starting positions between the top and bottom of the fabric (which can be encoded using a 1 and 0 respectively); the starting thread configuration completely controls the resulting pattern.

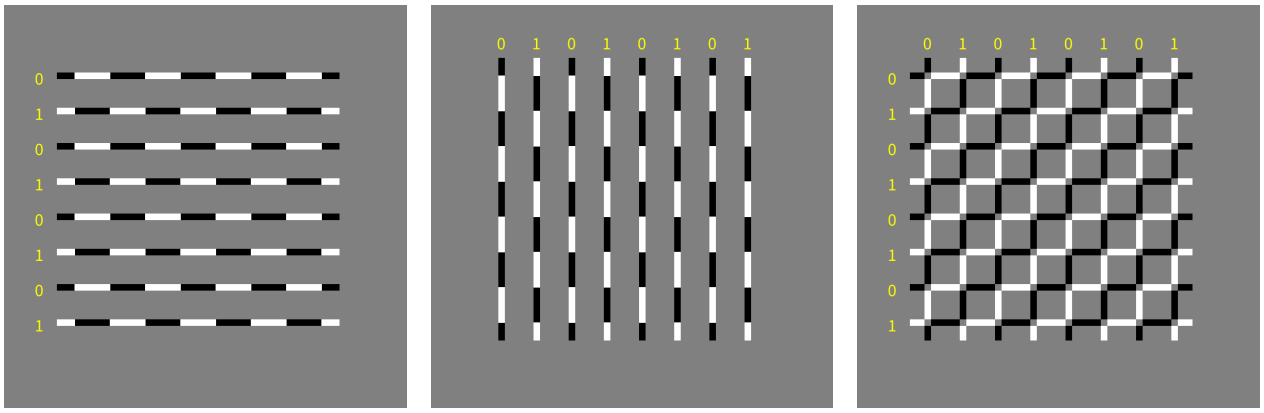
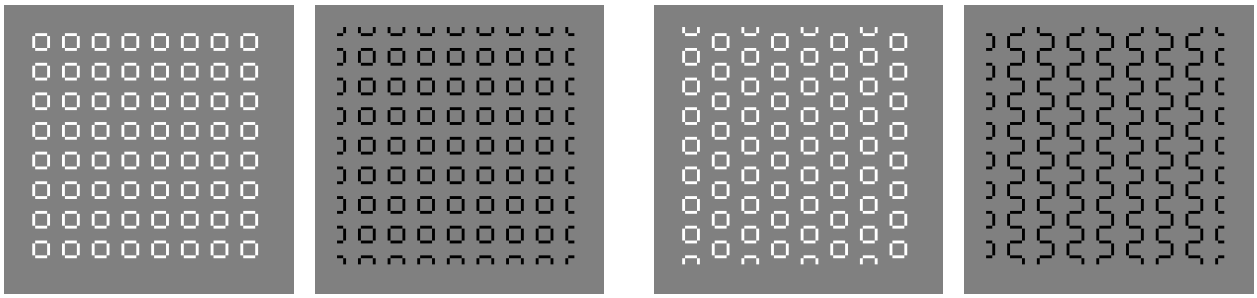


Figure 3: *Creating hitomezashi sashiko. Parallel running stitches are made along an orthogonal grid. Threads alternate between the top (white) or back (black) of the fabric. Connected patterns arise when the horizontal (left) and vertical (center) stitch patterns are combined (right). Stitches begin either on the bottom of the fabric (labeled 0) or on the top of the fabric (labeled 1).*

Hitomezashi and Truchet Connections

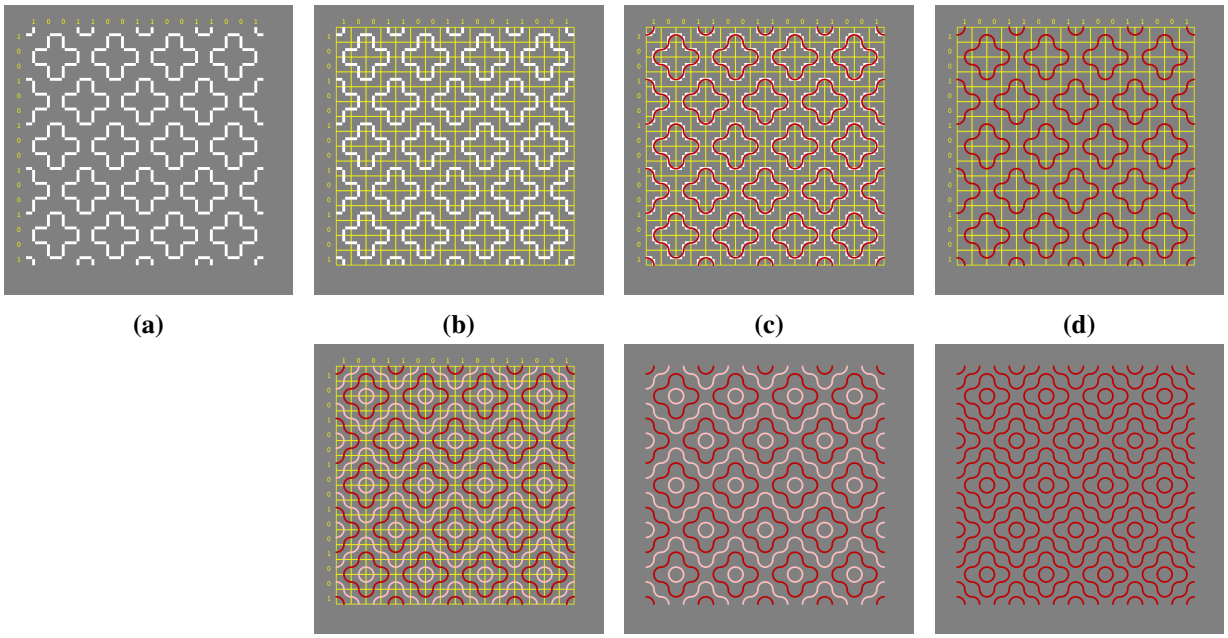
With hitomezashi, both sides of the fabric have related patterns that are complements of one another. Figure 3 shows how each row and column of stitch lies along a line segment that is completely covered. Thus when

the rows and columns are combined, all grid lines in the region are covered. Figure 4 shows two example patterns in white and their complementary back-side patterns in black.



(a) **(b)** **(c)** **(d)**

Figure 4: *Hitomezashi patterns are double-sided. The front-side pattern shown in (a) has the corresponding back-side pattern shown in (b), the patterns are translated versions of one another. The front-side pattern shown in (c) has the corresponding back-side pattern shown in (d).*



(a) **(b)** **(c)** **(d)**

(e) **(f)** **(g)**

Figure 5: *Transforming an example hitomezashi pattern into a corresponding Truchet pattern.*

Every hitomezashi pattern can be transformed into a corresponding Truchet tiling as illustrated with the example pattern in Figure 5. The first step is to see that the original hitomezashi pattern (Figure 5(a)) can be placed on a different grid where stitches run through grid centers rather than the grid edges (Figure 5(b)). A single full stitch now spans two grid cells. Each grid cell contains a 90° marking made from two half-stitches, one horizontal and one vertical, as seen in the upper-left-most square in Figure 5(b). This 90° marking can be replaced by a quarter-circle arc, shown in red, (Figure 5(c)) that is half of a Truchet tile (Figure 5(d)). The Truchet tiling can be completed, in pink, (Figure 5(e)); the pink curves correspond to the back-side hitomezashi pattern. The complete Truchet tiling without the grid is shown in (Figure 5(f)) and completely in red (Figure 5(g)).

Observations

While all hitomezashi patterns have a corresponding Truchet tiling counterpart, the converse is not true. Some, but not all, Truchet patterns have a hitomezashi counterpart. The number of Truchet patterns is always larger than the number of hitomezashi patterns on an $m \times n$ grid when m and n are both larger than 2. With a hitomezashi pattern, each of the m rows has two possible starting configurations, specifically the stitching begins either on the top of the fabric or the bottom of the fabric, resulting in 2^m possible patterns. Similarly, there are 2^n possible stitching patterns for the columns. Thus the total number of hitomezashi patterns is $2^m \times 2^n = 2^{m+n}$. For a Truchet pattern, each of the $m \times n$ tiles can be in one of two orientations, thus there are 2^{mn} possible Truchet patterns, which is greater than 2^{m+n} when m and n are both larger than 2.

A parity argument can be used to show that in any 2×2 block, a hitomezashi pattern's Truchet counterpart must have an even number of Truchet tiles oriented in each of the two possible directions. Let a, b, c, and d represent the thread configurations (0 or 1) coming into a 2×2 block. The orthogonal thread configurations give the 90° grid cell markings and thus the Truchet tile orientations. Stitches alternate as the thread advances from one cell to the next, which increments the stitch value by 1 (mod 2). Thus, for any 2×2 block we have

$$\begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} c & d \end{array} \\ \left[\begin{array}{c|c} a+c & (a+1)+d \\ \hline b+(c+1) & (b+1)+(d+1) \end{array}\right] \end{array}.$$

Summing the entries in the block gives $(a+c)+(a+1+d)+(b+c+1)+(b+1+d+1) = 2(a+b+c+d)+4$, which is always even. Thus there are always an even number of tiles for a given orientation.

The Truchet pattern (Smith variant) in Figure 1 (right) does not have a corresponding hitomezashi counterpart because the lower-right 2×2 block does not have even orientation parity because three tiles have one orientation and the fourth has the other orientation.

Conclusions and Future Work

One can overlay Truchet curves on a hitomezashi pattern such that half of the Truchet arcs follow the hitomezashi pattern and the other half of the Truchet arcs follow the obverse hitomezashi pattern. However, an arbitrary Truchet pattern will not always have a corresponding hitomezashi pattern. The connection between these two patterns may lead to new discoveries about one pattern type by applying knowledge from the other type.

References

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