Wire Construction of the Costa Surface and a Torus

Kodai Nakagawa[1], Tomohiro Tachi[2]

The University of Tokyo, Japan
[1]yoshikoisaburokoko0124@g.ecc.u-tokyo.ac.jp,
[2]tachi@idea.c.u-tokyo.ac.jp

Abstract

We show an approach for constructing grid patterns on the Costa surface and a torus with three points removed, which can be conformally mapped to each other. The orthogonal grids on the Costa surface and the torus are mapped from a common domain of a flat torus. When the slope of the mapped lines is rational, the generated orthogonal grid has finite length and can be constructed from two wire ropes orthogonally intersecting with each other.

Introduction

The Costa surface is a minimal surface of genus one with three boundaries [1] and is homeomorphic to a torus with three punctured holes. Costa surfaces have been manufactured in many ways and materials: for example, snow, bronze [2], and membrane [4] have been used to make models of the Costa surface. 3D printing is also a popular way to make it. This paper shows a novel way to make the Costa surface using wire ropes. Specifically, we show how to draw an orthogonal grid on the Costa surface using a conformal map between the Costa surface and a flat torus. We use the grid to fabricate the Costa surface using wire ropes. Due to its topology and the conformal property of the mapping, the grid on the Costa surface consists of only two curves, which means that the sculpture can be made with only two pieces of wire rope. The same method can be used to generate an orthogonal grid on the torus which can be conformally mapped to the Costa surface. Each orthogonal grid on the Costa surface and the corresponding grid on the torus have a two-parameter family. Moreover, we made one of them with wire ropes and metal fasteners.

Mapping a Line to the Torus and the Costa Surface

First, we consider a flat torus with a square fundamental domain, $T \coloneqq \mathbb{R}^2/\mathcal{L}$ (where $\mathcal{L} \coloneqq \{(m,n) \mid m,n \in \mathbb{Z}\}$), i.e., the unit square whose opposite boundaries are identified. Then, we consider a conformal embedding of T in $\mathbb{R}^3$, which is a torus of a specific proportion studied in [5]:

$$f\,:\,T \longmapsto \mathbb{R}^3, (u,v) \to \frac{(\cos 2\pi u\,, \sin 2\pi u\,, \sin 2\pi v)}{\sqrt{2} - \cos 2\pi v}$$

The Costa surface can be described as a conformal mapping from a flat torus, as described in [2]:

$$\begin{aligned} g\,:\,T \longmapsto \mathbb{R}^3, (u,v) \to (&\frac{1}{2}\mathrm{Re}\left\{-\zeta(u+iv) + \pi u + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\zeta\left(u+iv-\frac{1}{2}\right) - \zeta\left(u+iv-\frac{i}{2}\right)\right]\right\}, \\ &\frac{1}{2}\mathrm{Re}\left\{-i\zeta(u+iv) + \pi v + \frac{\pi^2}{4e_1} - \frac{\pi}{2e_1}\left[i\zeta\left(u+iv-\frac{1}{2}\right) - i\zeta\left(u+iv-\frac{i}{2}\right)\right]\right\}, \\ &\frac{1}{4}\sqrt{2\pi}\ln\left|\frac{\wp(u+iv)-e_1}{\wp(u+iv)+e_1}\right|), \end{aligned}$$

where ζ is the Weierstrass zeta function, $\wp$ is the Weierstrass elliptic function whose half periods are 1/2 and i/2, and $e_1 \coloneqq \wp(1/2) \approx 6.88$.

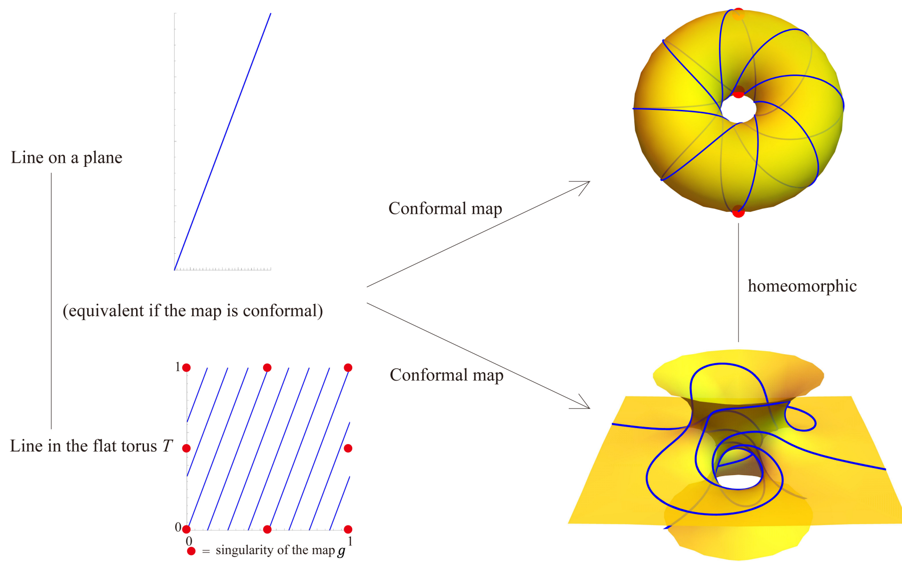


Figure 1: *Mapping a line onto the torus and the Costa surface.*

The map $g : T \longmapsto \mathbb{R}^3$ is not defined at (0,0), (0,1/2), (0,1), (1/2,0), (1/2,1), (1,0), (1,1/2) and (1,1), and $g(u, v)$ diverges around these singularities. Therefore, the result of mapping also diverges when the initial line passes through the singularities.

Now consider mapping a line on a plane (which can be written as $v = au + b$) onto the torus and the Costa surface. Because of the double periodicity of the quotient map $\mathbb{R}^2 \to T$, this gives repeating patterns on the flat torus T and thus the embedded torus and the Costa surface, as shown in Figure 1. Note that this curve does not self-intersect.

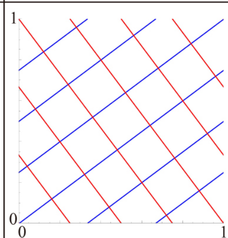
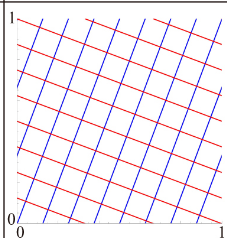
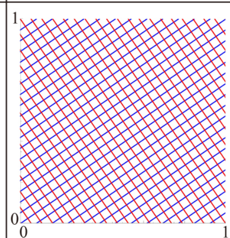
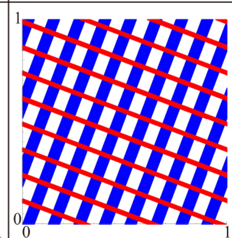
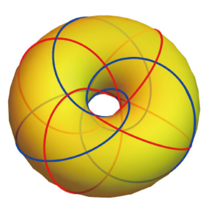
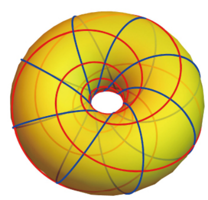
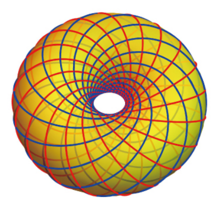
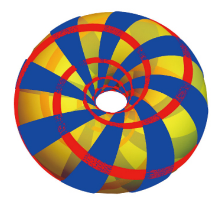
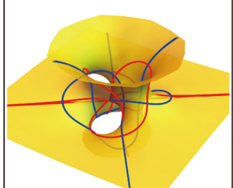
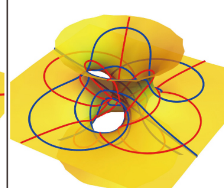
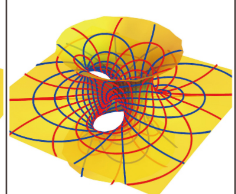
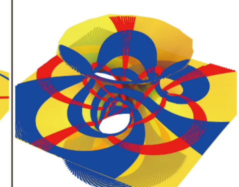
If the slope a of the initial line is rational (i.e., $a = p/q$ where p and q are coprime integers), the line in T goes back to the starting point because the line $v = (p/q)\, u$ passes through (0,0) and (q, p) belonging to $\mathcal{L}$. The corresponding curve on the torus is also closed in finite length and forms a (p, q) torus knot, i.e., the curve on the torus winds around the torus p times in the direction of the meridian and q times in the direction of the longitude. In the same way, the resulting curve on the Costa surface is closed if a is rational. Here, we can choose b such that the repeated lines in T do not pass through the singularities.

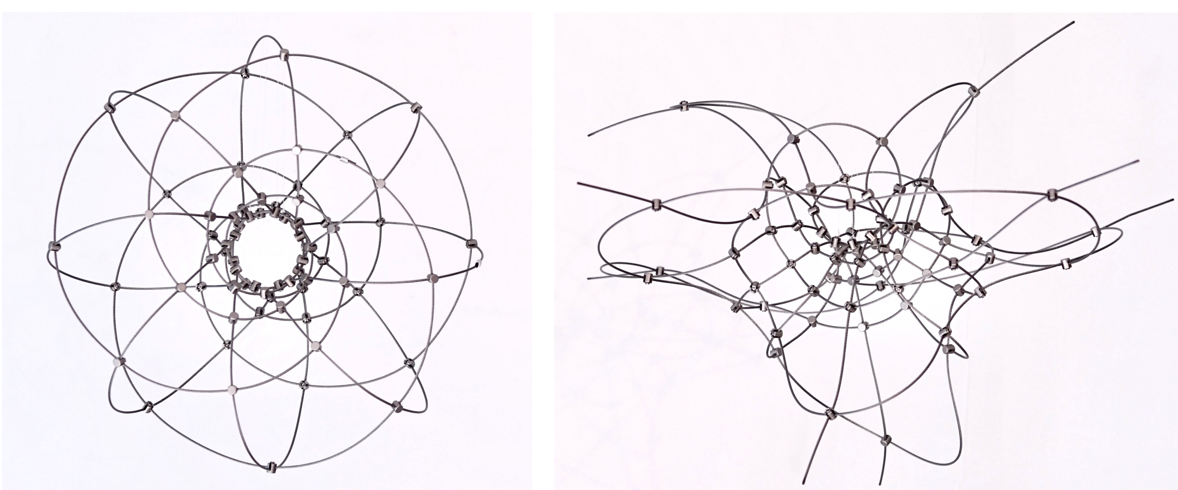
When a is irrational, the line in T does not go back to where it starts. This is because the line on a plane $v = au$ does not pass through any point in $\mathcal{L}$ except (0,0) when $b = 0$. Therefore, the curve on the Costa surface or the torus also travels around on the surface infinitely and densely covers the surface. Interesting results are obtained when the value of a is close to a simple rational number (e.g., $2^{\sqrt{2}} = 2.66514 \approx 8/3$). The curve on the Costa surface or the torus slightly moves aside every time it goes round, and it forms what seems like a band on the surface when we limit the total length of the curve.

Mapping an Orthogonal Grid and Fabricating the Costa Surface

The curve mentioned above does not stand alone if it is made with wire ropes. So, the next step is to add another curve to make it stable. Because the fastener we used was to fix two pieces of wire rope to cross at right angle, we decided to construct two curves intersecting with each other orthogonally. Adding one line which crosses at right angle with the initial line $v = au$ (specifically, $v = (-1/a)\, u$) is enough to generate an orthogonal grid in the flat torus T. This grid appears as two curves on the Costa surface or the torus when mapped. Thanks to the conformal property of the map, the orthogonality of the grid is conserved, i.e., the two generated curves on the Costa surface or the torus intersect with each other many times and always orthogonally. Some of the variations of the grid on the surfaces are shown in Table 1. The number of the intersecting points of the two curves is the number of wire rope fasteners that need to be prepared. If a is rational, the area of the unit cell of the grid in the flat torus T is $1/(p^2 + q^2)$, so the number of the unit cells in T is $p^2 + q^2$. Therefore, by the Euler characteristic formula for torus, the number of the intersecting points is also $p^2 + q^2$.

Table 1: *The Grids on the Surfaces with Different Values of a.*

value of a	3/4	8/3	13/19	$2^{\sqrt{2}}$
grid in T				
grid on the torus				
grid on the Costa surface				



(a) (b)

Figure 2: *(a) The finished model of the torus, and (b) the finished model of the Costa surface.*

We chose 8/3 as the value of a for actual construction. Some parts of the curves generated on the Costa surface are too big relative to other parts to be fabricated, so such parts are cut away. Actually, four wire-lobes were cut away, and as a result, the number of curves on the Costa surface went from two to four. Prior to the manufacture, the curves are scaled to a size suitable for production and exhibition. The curves on the Costa surface or the torus are made by hand with wire ropes. The lengths of the wire ropes between intersecting points were calculated in advance using a parametric CAD system, Rhinoceros and Grasshopper. The wire ropes are fixed to each other at right angle at all intersecting points using a fastener (SCP-1, Arakawa & Co., Ltd). Ideally, the number of fasteners is $3^2 + 8^2 = 73$, but the actual number of fasteners we used was 72 because an intersecting point was in a part which was cut away. The minimum distance between two adjacent fasteners is 30 milimeters, which is long enough compared to the diameter of the fastner (10 milimeters). The bending elasticity of the wire naturally led to the smooth appearance of the torus (Figure 2(a)) and the Costa surface (Figure 2(b)); they look like the skeletons of the corresponding surfaces. A video of the sculptures rotating is attached as a supplement to clearly show the three-dimensional configuration of the wire ropes.

By extending our approach, it is also possible to generate a kagome pattern, (i.e. triaxial weaving, featured in artworks such as [3]) on the Costa surface and the torus (Figure 3). However, it needs some approximation because the ratio of the slopes of the lines in a regular kagome pattern is irrational, and such a pattern cannot be generated by the same method. Mapping three lines v =(13/3)u (shown in red in Figure 3), v =(5/14)u (shown in blue), v =(-8/11)u-1/22 (shown in green) gives a reasonable approximation of a regular kagome pattern.

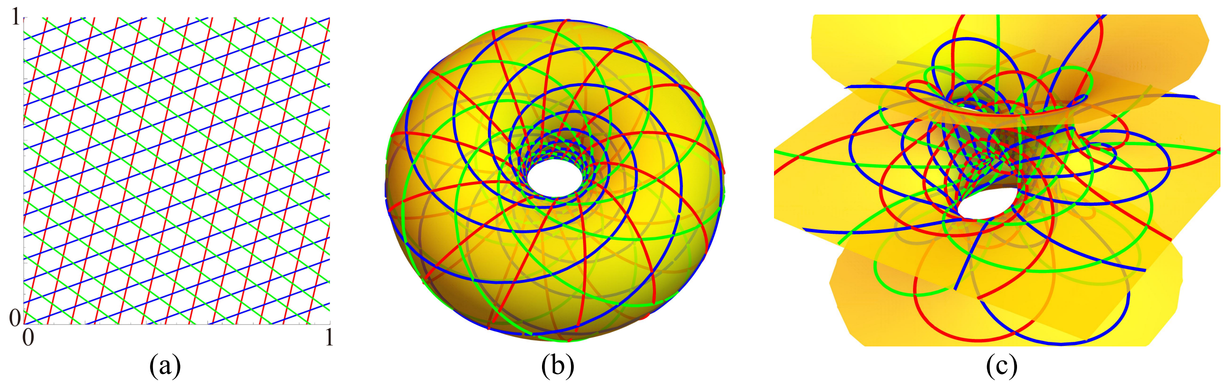


Figure 3: *Generated kagome patterns: (a) in the flat torus T, (b) on the torus, (c) on the Costa surface.*

Summary and Conclusions

We showed a family of orthogonal grids on the Costa surface and the torus that can be conformally mapped from a flat torus with the square fundamental domain. We used this idea to make a novel wire-rope representation of the Costa surface and the torus. The sculptures demonstrate the topological relation between the Costa surface and a torus.

References

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