Constructing Sierpinski Tetrahedrons from Connector Pieces

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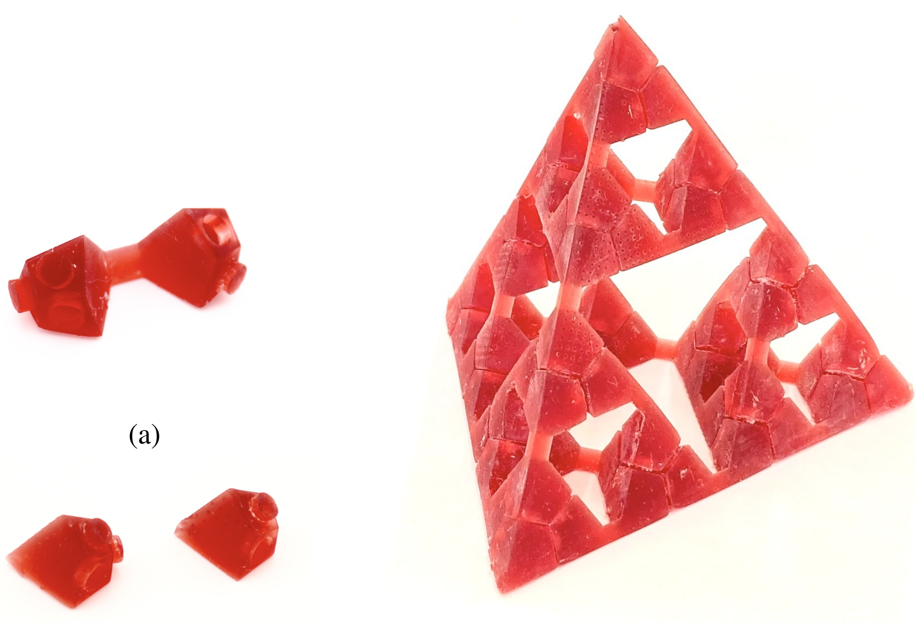
Abstract

We present a construction method for Sierpinski tetrahedron objects that does not consider individual tetrahedrons as basic building blocks but rather regards connectors of two tetrahedrons as the fundamental units. It uses pieces with an identical shape that is a union of two fragments of regular tetrahedrons connected at their vertices, and it has the property that one can only build finite approximations of the Sierpinski tetrahedron if four corner pieces are supplemented. We have 3D-printed two sets of pieces that have different mechanisms to link the pieces. One uses 3D-printed joints and it can be used for constructing Sierpinski tetrahedron objects. The other one uses magnets and it can also be used as a puzzle. We describe how one would construct Sierpinski tetrahedron objects by solving this puzzle, even without any prior knowledge about Sierpinski tetrahedrons.

Introduction

The Sierpinski tetrahedron is a well-known three-dimensional fractal shape with many fascinating properties such as square projections [2][3], and it is an effective educational resource for students from elementary school to university level. In particular, building physical models of approximate Sierpinski tetrahedrons is



(a) (b) (c)

Figure 1: *(a) A basic piece, (b) corner pieces, (c) an object constructed from 30 basic pieces and 4 corner pieces.*

an engaging activity, and it is popular in many workshops and classes [1]. To construct a model, one would first assemble regular tetrahedrons and then repeat connecting four of the products from the previous stage to produce the next one. This activity to create increasingly large structures is highly enjoyable and provides a tangible way to comprehend recursive procedures. However, connecting two polyhedrons at their vertices can be difficult, and using tape to do so results in an object that is not only weak, but also unattractive.

In this paper, we explore a method of constructing approximate Sierpinski tetrahedron objects that does not use regular tetrahedrons as fundamental pieces but rather utilizes connectors between two tetrahedrons as fundamental objects. From $2 \times 4^n - 2$ copies of the basic piece in Figure 1(a) and two copies of the corner piece of each type in Figure 1(b), one can construct a level-n approximation model of the Sierpinski tetrahedron (Figure 1(c) for the case $n = 2$). Furthermore, one can show that they are the only shapes that can be constructed from the pieces.

We have 3D-printed two sets of pieces that have different connector mechanisms. One (Figure 1) uses 3D-printed joints and it can be used for constructing Sierpinski tetrahedron objects. The other set (Figure 9) uses magnets and it can also be used as a puzzle. This puzzle not only provides entertainment but also serves as a mathematical learning tool; since approximation models of the Sierpinski tetrahedron are the only objects that can be constructed from the pieces, a person without prior knowledge would inevitably construct them.

A Basic Piece and Constructions of some Fundamental Shapes

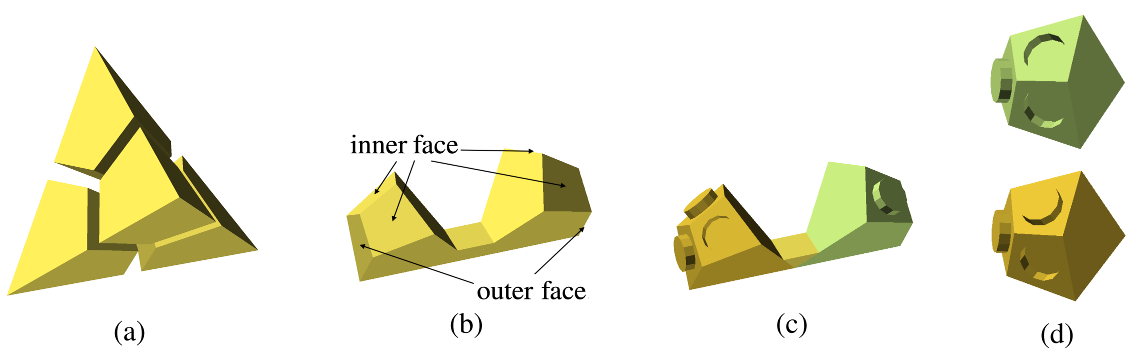


Figure 2: *(a) Four hexahedrons that form a regular tetrahedron, (b) a basic piece, (c) a basic piece with polarities, (d) corner pieces.*

A regular tetrahedron can be divided into four congruent hexahedrons (Figure 2(a)). By connecting two of these at the vertices of the tetrahedrons, we get a shape in Figure 2(b). We call the six congruent kite faces that are cross sections of tetrahedrons the connector faces. Connector faces are divided into inner and outer faces as in the figure. We assign polarities to them, making two outer faces positive and negative and the four inner faces alternating positive and negative. To visualize them, we add a cylinder to positive faces, remove a cylinder from negative faces, and use different colors as in Figure 2(c). We add a strut that connects the hexahedrons, but it is for visualization and we ignore it in mathematical treatments. We call such an object a *basic piece* and the two hexahedrons *heads* of a basic piece. The object in Figure 1(a) is a model of a basic piece. It is important to note that a basic piece is a chiral, i.e., it cannot be superposed on its mirror image by any combination of rotations and translations. We fix one chirality, and its mirror image is not considered as a basic piece.

In addition to basic pieces, we consider two kinds of corner pieces in Figure 2(d) that are hexahedrons with polarities added to three kite faces; one has two positive faces and the other has two negative faces. We consider objects constructed from these two kinds of pieces by gluing positive and negative faces. We call such an object a *shape*, and a shape without an open connector face a *closed shape*. We study what kind of closed shapes can be constructed using some basic pieces and two corner pieces of each kind. In the following, we simply refer to a closed shape constructed from such a set of pieces as a closed shape.

In order not to leave open connector faces, if two pieces are connected then two more pieces must be connected to them so that the heads of the four pieces form a regular tetrahedron. From this observation, it is clear that a closed shape is a union of congruent regular tetrahedrons that are connected at vertices. In addition, all of the regular tetrahedrons must have parallel faces.

For the study of closed shapes, the existence of corner pieces complicates the arguments. Therefore, we start our observation with how the corner pieces are used. Since a closed shape consists of regular tetrahedrons with parallel faces, one can consider the minimal tetrahedron M that contains the shape and has faces parallel to the regular tetrahedrons. Such a tetrahedron M is a regular tetrahedron. Note that whether a vertex of a component tetrahedron is connected to another tetrahedron is determined by whether that vertex is a part of a basic piece or a corner piece. Therefore, for each vertex direction, there is only one vertex of one tetrahedron that is not connected to another tetrahedron since there are only four corner pieces. It means that the convex hull of the shape is the tetrahedron M, and the corner pieces are located at the four vertices of M.

We first study arrangements of four basic pieces whose heads form a regular tetrahedron. If every outer face of the four heads is connected to an inner face of another head, then three of the pieces are connected as in Figure 3. In these cases, one cannot connect another piece to form a regular tetrahedron because the polarities of the three connector faces are the same.

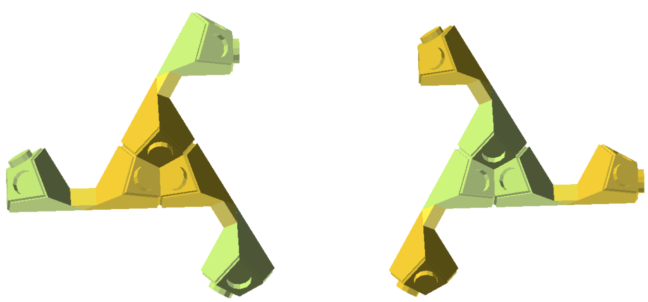
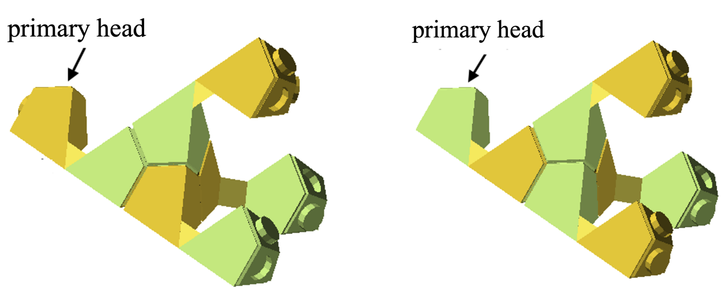


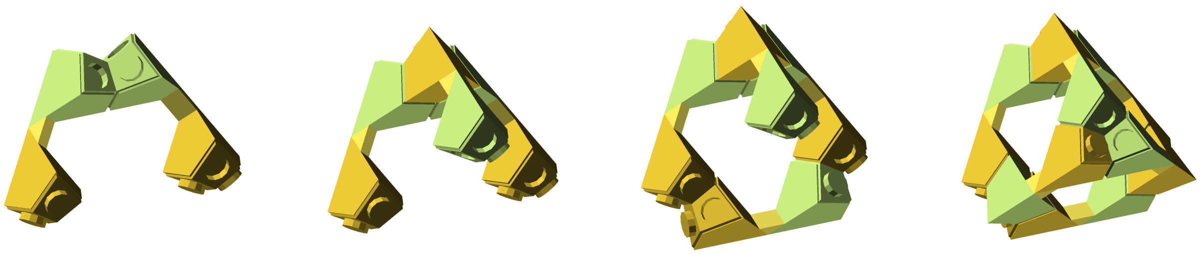
Figure 3: *Inappropriate arrangements of three basic pieces.*



(a) (b)

Figure 4: *(a) Positive* S_0*-shape, (b) negative* S_0*-shape.*

Therefore, in order for four basic pieces to create a tetrahedron, two of the pieces must be connected outer face to outer face. There are only two such shapes modulo congruence that are depicted in Figure 4 (a) and (b). We call them S_0-*shapes*. Each S_0-shape consists of a regular tetrahedron and four heads. Among the four heads, three are positioned above the same face of the tetrahedron, while the other one is above a different face. We call the latter head the *primary head* of the S_0-shape. In Figure 4, the primary head of (a) has a positive outer face and that of (b) has a negative outer face. Therefore, we call them *positive* and *negative* S_0-*shapes*, respectively.

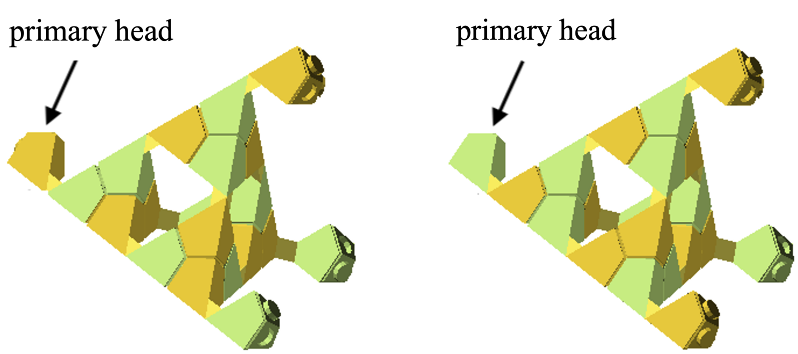


(a) (b) (c) (d)

Figure 5: *Construction of an octa-shape.*

Next, we consider the case where heads of two basic pieces are connected inner face to inner face as in Figure 5(a) and study how these pieces can be extended in order to form a closed shape. To their other inner faces, a corner piece cannot be connected as we have observed. Thus, yet another basic piece must be connected and Figure 5(b) is the unique arrangement that fulfills the polarity condition. This shape has three open heads, and each pair of them can be connected by a basic piece via inner-faces as in Figure 5(c). If we try to connect basic pieces to these inner faces of the two heads, then collision of pieces will occur and Figure 5(c) is the unique possible arrangement. In this way, six pieces are forced to be connected as in Figure 5(d). We call such a shape with an octahedral open space an *octa-shape*, and the four incomplete tetrahedrons of an octa-shape *open nodes*.

It is important to note that each piece of an octa-shape can be rotated independently by 180 degrees to form an octa-shape with different polarities of the three faces at their open nodes. Among such configurations, those with polarities $(+,+,+)$ and $(-,-,-)$ should be excluded because no piece can be connected. Thus, the possible polarities of the four open nodes are $(+,-,-)$, $(+,-,-)$, $(+,+,-)$ and $(+,+,-)$.

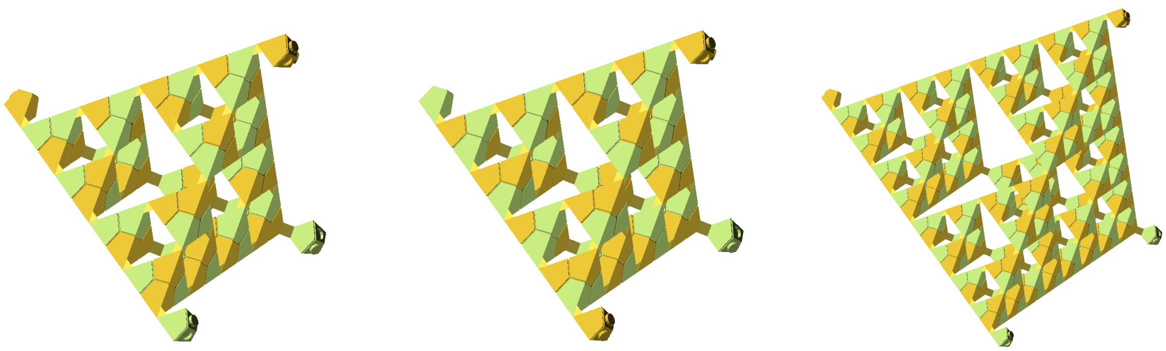


(a) (b)

Figure 6: *(a) Positive* S_1*-shape, (b) negative* S_1*-shape.*

By connecting four basic pieces to the open nodes of an octa-shape, one can form the first approximation of the Sierpinski tetrahedron extended with four heads (Figure 6). We call such shapes S_1-*shapes*. Since the two outer faces of a basic piece have opposite polarities, one can conflate the S_1-shape into a S_0-shape by removing the interior octa-shape and connecting the four heads. Conversely, an S_1-shape can be created from a S_0-shape by deconstructing the tetrahedron and inserting an octa-shape. Note that the way the six basic pieces are inserted is unique and therefore the correspondence between S_1-shapes and S_0-shapes is one-to-one. Thus, there are two kinds of S_1-shapes. An S_1-shape has four heads, and we define the primary head of an S_1-shape and positive and negative S_1-shapes (Figure 6(a, b)) as we did for S_0-shapes.

An S_1-shape contains four S_0-shapes: two positive and two negative S_0-shapes. The primary head of each S_0-shape is a head of the S_1-shape, and the other three heads are used to connect it to other S_0-shapes. Therefore, one can construct an S_1-shape from two positive and two negative S_0-shapes, by removing six overlapping non-primary heads and connecting them. It is the case both for positive and negative S_1-shapes. One can do the same construction from four S_1-shapes to form the second approximation of the Sierpinski tetrahedron extended with four heads (Figure 7(a, b)). We call such shapes S_2-shapes. With the same arguments, there are two kinds of S_2-shapes that we call positive and negative S_2-shapes.



(a) (b) (c)

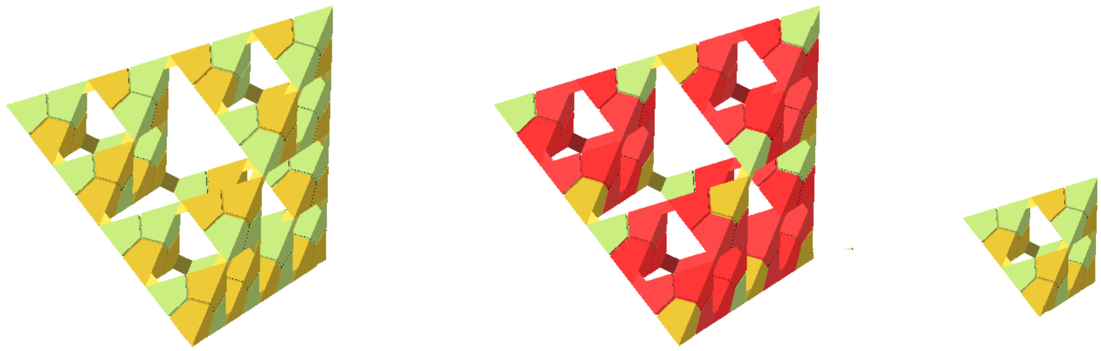
Figure 7: *(a) Positive* S_2*-shape, (b) negative* S_2*-shape, (c) positive* S_3*-shape.*

In the same way, we can construct an S_{n+1}-shape from four S_n-shapes for $n \geq 0$. An S_n-shape is the level-n approximation of the Sierpinski tetrahedron extended with four heads. There are two kinds of S_n-shapes that we call positive and negative S_n-shapes. Note that the two S_n-shapes differ in only the directions of the $2^n + 1$ pieces on the edge starting with the primary head, and one can switch between the two shapes by rotating these $2^n + 1$ pieces.

Closed Shapes

From an S_n-shape, one can create the level-n approximation of the Sierpinski tetrahedron by replacing the four basic pieces at the corners with four corner pieces (Figure 8(a) for the case $n = 2$). We show that approximations of the Sierpinski tetrahedrons are the only closed shapes.

As we observed, in a closed shape, (1) if two basic pieces are connected inner face to inner face, then they are parts of an octa-shape, (2) if two basic pieces are connected inner face to outer face, then the latter piece is contained in an octa-shape and both heads of the former piece are connected to octa-shapes. Thus, every closed shape is constructed by connecting open nodes of octa-shapes by using basic pieces and finally closing four open nodes with corner pieces as Figure 8(b) shows for the case of the second approximation of the Sierpinski tetrahedron constructed from a positive S_2-shape. From this observation, we have the following lemma (see Figure 8(b, c) for an example).



(a) (b) (c)

Figure 8: *(a) Second approximation of the Sierpinski tetrahedron, (b) octa-shapes, (c) first approximation of the Sierpinski tetrahedron obtained by removing octa-shapes.*

Lemma 1. (1) From a closed shape, one can form another closed shape by deconstructing all its tetrahedrons and inserting octa-shapes.

(2) From a closed shape that is not a regular tetrahedron (i.e., one composed of four corner pieces), one can form another closed shape by removing all the octa-shapes and connecting the four pieces that were connected to the same octa-shape.

(3) The two operations in (1) and (2) are inverses of each other.

Thus, by repeatedly applying Lemma 1(2) to a closed shape, we have smaller and smaller closed shapes and we finally obtain a regular tetrahedron. Since the operations of Lemma 1(1) and Lemma 1(2) are inverses of each other, one can form all the closed shapes by repeatedly applying the operation of Lemma 1(1) to a regular tetrahedron. Therefore, we have the following.

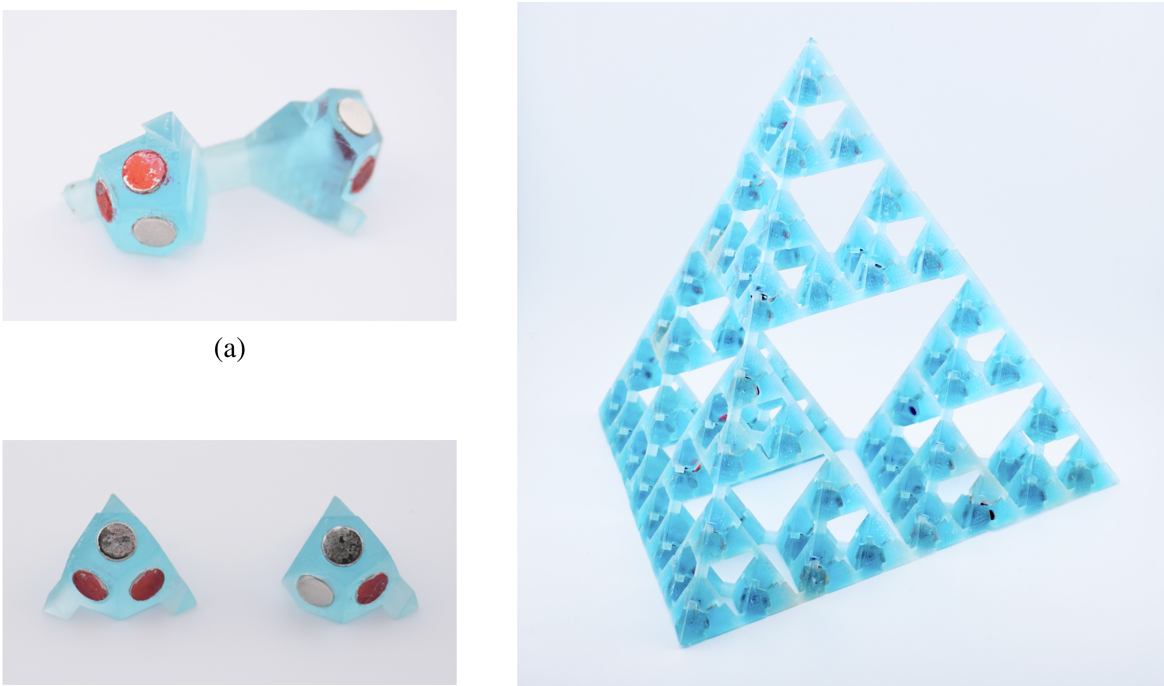
Theorem 2. All of the closed shapes are finite approximations of the Sierpinski tetrahedron.

As we have observed, there are two S_n-shapes modulo congruences for every $n \geq 0$. Therefore, there are two ways of arranging pieces to form an n-th Sierpinski object modulo congruences. As was the case for S_n-shapes, the two arrangements of the pieces are identical except for the directions of the pieces on one edge. Note that both of the arrangements do not have rotational symmetry, which is verified only through the arrangements of corner pieces. Therefore, by multiplying 2 with the order of rotational symmetry of a n-th Sierpinski object, there are 24 possible arrangements of the pieces of an n-th Sierpinski object.

A Sierpinski Tetrahedron Puzzle

We made the pieces using a 3D printer in two different ways. One set of pieces is shown in Figure 1(a,b) and their shapes are depicted in Figure 2(c,d). With this connector, one needs to connect four connectors simultaneously to form a tetrahedron node and it is not an easy process. Therefore, it is not suitable for finding the correct structure through trial and error. However, it firmly connects pieces and therefore works well for making Sierpinski tetrahedron objects.

The other one is shown in Figure 9(a,b) and it uses magnets to connect pieces. In addition to magnets, each piece has sticks on positive faces and corresponding hollows on negative faces so that two connector



(a) (b) (c)

Figure 9: *(a) A basic piece with magnets, (b) corner pieces with magnets, (c) third level approximation of the Sierpinski tetrahedron.*

faces with different polarity stick together only in the correct way. This magnet mechanism makes it easy to assemble and disassemble objects, which is important for a puzzle or a toy.

As we studied, all of the closed objects constructed from these pieces are finite approximations of the Sierpinski tetrahedron. Therefore, solving the puzzle involves constructing Sierpinski tetrahedron objects. In particular, one can play with this puzzle without knowing what it will lead and discover the shape for themselves.

Most people form an octa-shape while randomly connecting the pieces. However, one will get stuck if the polarity of an open node is $(+,+,+)$ or $(-,-,-)$; trial and error will quickly reveal the need to rotate pieces to change the polarity, and inevitably an S_1-shape will emerge. As the next step, one naturally proceeds with the construction of an S_2-shape by extending the three non-primary heads. Here, the same difficulty may occur in each of the three additional S_1-shapes, and moreover, one needs to arrange the four shapes so that they fit together to form an S_2-shape. Note that there are four ways of extending a given S_1-shape into a S_2-shape as Figure 10 shows for the case of a positive S_1-shape. In this figure, (b) and (c) are positive whereas (d) and (e) are negative S_2-shapes, and the primary head of (a) becomes the primary head only in (b). One would naturally intend to assemble (b), but finally one of these four configurations may be obtained. Then, one will recognize that it also has three non-primary heads and can proceed to the next level. The next step would be more complicated but one could think of the procedure from the recursive structure, and a third level approximation object is obtained from 126 basic pieces and four corner pieces (Figure 9(c)).

In this process, one can concentrate on the puzzle and does not need to care about irrelevant things like cutting sticky tapes. In addition, one can correct errors easily, and enjoy the construction repeatedly. The beauty of the final object is so impressive and the author believes that this puzzle gives a good mathematical experience to the player.

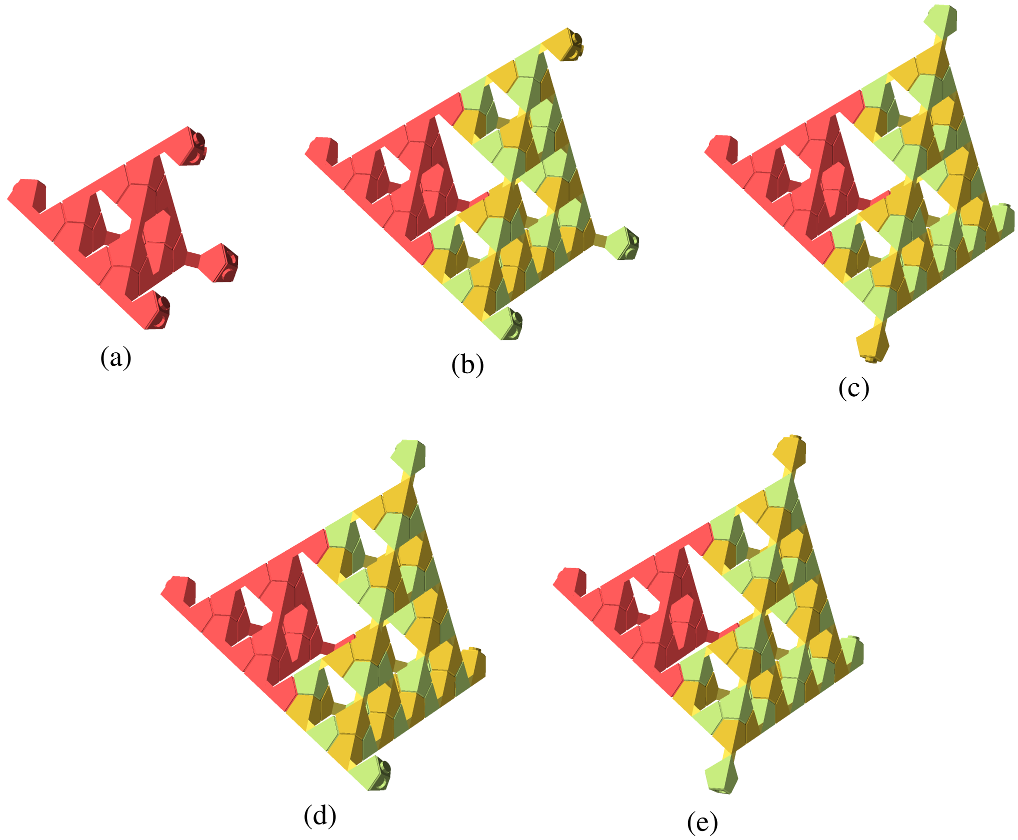


Figure 10: *(a) A positive* S_1*-shape, (b, c, d, e) four possible extensions to* S_2*-shapes.*

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