Using Triangle Sierpinski Relatives to Visualize Subgroups of the Symmetries of the Square

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Abstract

The Sierpinski relatives form a fascinating class of fractals because they all possess the same fractal dimension but can look very different. The famous Sierpinski gasket is one of these relatives. The convex hull of the gasket has a boundary that is a right isosceles triangle. There is a sub-class of relatives that all have the same convex hull as the gasket, and are referred to as triangle relatives. The triangle relatives can be used to build other beautiful fractals. In particular, one can build fractals with square convex hulls and these can be used to visualize subgroups of the symmetries of the square.

Introduction

The Sierpinski gasket is a well-known fractal whose boundary is a triangle (either equilateral or right isosceles) [6]. This paper focuses on the right isosceles version, see Figure 1(a), because of its connections with the symmetries of the square. One way to generate the gasket is with an iterated function system (IFS) [1]. The IFS of the gasket can be modified using the symmetries of the square. This yields an interesting class of fractals, called the Sierpinski relatives, that all have the same fractal dimension but different topologies [2,4,7]. Figures 1(b), (c) and (d) display three relatives. My research includes using methods of computational topology to classify and characterize fractals, with the main goal of getting beyond fractal dimension. I initially thought the relatives would be a good summer project, and I am still working on them more than 15 years later. I found that the convex hulls of the relatives can help with the computational topology analysis, and these convex hulls all have polygonal boundaries [10]. In particular, there is a sub-class for which the polygonal boundaries are the same right isosceles triangle as the boundary of the gasket. Such triangles can be used to form squares in different ways, and thus I was inspired to create new fractals by joining together these 'triangle' relatives.

I often teach a course in abstract algebra, and one of my favourite examples of a group is the symmetries of the square [3]. I encourage students to have a square in front of them so they can explore how the symmetries work. I am often looking for creative ways to help my students visualize the subgroups of the symmetries and to see how powerful the symmetries are. I also like to bring my own research into the classroom, so the relatives are wonderful because they are generated using the symmetries of the square and can be used to build other fractals that have square convex hulls.

(a) (b) (c) (d)

Figure 1: *Sierpinski Gasket and three Sierpinski relatives.*

Mathematical Background

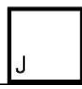
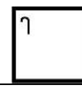
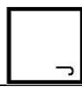
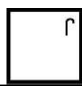
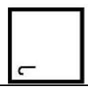
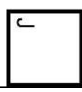
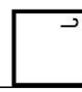
Groups, Subgroups, and the Symmetries of the Square

A *group* [3] is a set of elements G with a binary operation $*$ such that $x * y \in G$ for all $x, y \in G$ and

- The operation is *associative*: $x * (y * z) = (x * y) * z$ for all $x, y, z \in G$;
- There is an *identity* element e such that $x * e = x = e * x$ for all $x \in G$;
- Every element x has an *inverse* y such that $x * y = e = y * x$.

The set of integers with addition is a group: the identity is 0; the inverse of x is $-x$.

Table 1: *Description of the symmetries of the square.*

Label	Action	Verbal Description	Label	Action	Verbal Description
a (ρ_0)		No change	e (μ_2)		Reflection across horizontal line
b (ρ_1)		90° rotation counter-clockwise	f (μ_1)		Reflection across vertical line
c (ρ_2)		180° rotation	g (δ_1)		Diagonal Reflection
d (ρ_3)		270° rotation	h (δ_2)		Oher Diagonal Reflection

The *nth dihedral group* D_n consists of the set of symmetries for a regular polygon with n sides under composition [3]. D_n has $2n$ elements: n rotations and n reflections. The *symmetries of the square*, D_4, are described in Table 1. To simplify the labelling of the Sierpinski relatives (and to be consistent with previous papers by the author), the notation differs from Fraleigh (given in parentheses) [3]. The colours are used to identify the individual symmetries but also to distinguish between rotations and reflections. The action of a symmetry on a square is demonstrated by showing how the "J" moves under the given symmetry.

*	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	b	c	d	a	g	h	f	e
c	c	d	a	b	f	e	h	g
d	d	a	b	c	h	g	e	f
e	e	h	f	g	a	c	d	b
f	f	g	e	h	c	a	b	d
g	g	e	h	f	b	d	a	c
h	h	f	g	e	d	b	c	a

$\{a, b, c, d, e, f, g, h\}$

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

$\{a, b, c, d\}$

*	a	c	e	f
a	a	c	e	f
c	c	a	f	e
e	e	f	a	c
f	f	e	c	a

$\{a, c, e, f\}$

*	a	c	g	h
a	a	c	g	h
c	c	a	h	g
g	g	h	a	c
h	h	g	c	a

$\{a, c, g, h\}$

*	a	c
a	a	c
c	c	a

$\{a, c\}$

*	a	e
a	a	e
e	e	a

$\{a, e\}$

*	a	f
a	a	f
f	f	a

$\{a, f\}$

*	a	g
a	a	g
g	g	a

$\{a, g\}$

*	a	h
a	a	h
h	h	a

$\{a, h\}$

Figure 2: *Binary operation tables for the symmetries of the square and non-trivial subgroups.*

Figure 2 displays a binary operation table for D_4, where $x * y$ means do y first, then do x. Each entry is of the form row∗column. For example, rotation by 90° followed by vertical reflection is the same as a diagonal reflection: $f * b = g$. Note that the order matters: $b * f = h$. The identity is a, and every element has an inverse. For example, the inverse of b is d because they combine to give a rotation of 360°.

A group is *abelian* if the binary operation is commutative ($x * y = y * x$ for all $x, y \in G$). Otherwise a group is said to be *non-abelian*. The dihedral groups D_n, for $n \geq 3$, are non-abelian [3]. A *subgroup* of a group is a subset that is also a group under the same operation. Every group has itself and the set containing only the identity as subgroups. The set of even integers forms a subgroup of the integers with addition because the sum of any two even integers is even. The odd integers do not form a subgroup because the sum of any two odds is not odd. The tables for the non-trivial subgroups of D_4 are displayed in Figure 2.

Iterated Function Systems, Sierpinski Relatives, and Convex Hulls

An *iterated function system* (IFS) is a collection of contractive mappings $\{f_1, f_2, \ldots, f_n\}$. A given IFS has a unique attractor A that satisfies $A = f_1(A) \cup f_2(A) \ldots f_n(A)$ [1,4]. Thus A is made of smaller versions of itself. Starting with any compact set X, form a sequence of approximations $\{A_n\}$, for $n \geq 0$, as follows. $A_0 = X$ and for $n \geq 1$:

$$A_n = \cup_{i=1}^{n} f_i(A_{n-1}) = f_1(A_{n-1}) \cup f_2(A_{n-1}) \cup \ldots \cup f_n(A_{n-1}).$$

The limit of the approximations as $n \to \infty$ is A [1,4].

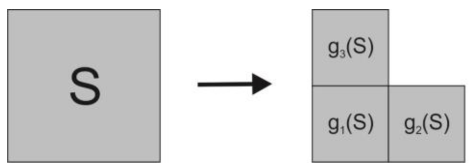


Figure 3: *The unit square and result of applying maps* g_1, g_2, g_3.

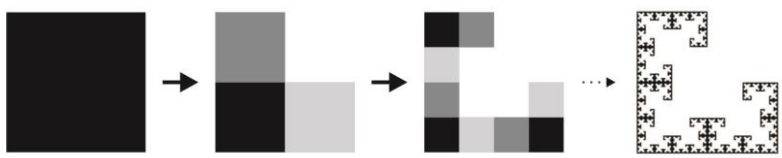


Figure 4: *Maps to obtain the relative* R_{abd}.

A *Sierpinski relative* R_{xyz}, where x, y, z are elements of D_4, has IFS $\{g_1, g_2, g_3\}$ as follows [2,4,7]. Let S be the unit square with vertices (0,0), (1,0), (1,1) and (0,1). All three maps apply a symmetry (x for g_1, y for g_2, z for g_3), then a contraction by a factor of 2. The map g_2 then shifts the contracted square to the right by ½ and the map g_3 shifts the contracted square up by ½, see Figure 3. The gasket is the unique attractor of the IFS where all maps preserve the orientation of the square, so R_{aaa}. The relative R_{abd} and the first three approximations are shown in Figure 4. Figures 1(b), (c), and (d) display other relatives.

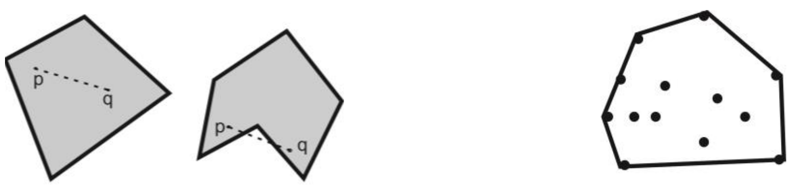
A relative is self-similar because it is made from smaller versions of itself without gaps or overlaps. The similarity dimension D of a self-similar object satisfies $Nr^D = 1$, where N is the number of smaller versions and r is the scaling ratio [6]. In the case of the relatives, $N = 3$ and $r = 1/2$. Thus $D = \ln 3 / \ln 2 \approx 1.585$.

There are 8 choices for each symmetry of the square, thus there are $8^3 = 512$ possibilities for the IFS. The attractors are not all distinct; there are 232 distinct attractors [8,9]. The gasket is symmetric via the diagonal symmetry g, thus also corresponds to $R_{aag}, R_{aga}, R_{gaa}, R_{agg}, R_{gag}, R_{gga}$ and R_{ggg}. The map g is the only possible non-trivial symmetry for a relative. A previous paper explored the symmetric relatives

[11]. The only non-trivial isometry between two relatives is the map g [9]. Non-symmetric relatives come in congruent pairs. Two relatives R and R' are congruent if and only if $R' = g(R)$. If R is not symmetric, one can obtain the unique *congruent match* to R[9]. Let $R = R_{xyz}$ be such that $R \neq g(R)$. Then $R' = R_{uvw}$, where $u = gxg, v = gzg, w = gyg,$ is congruent to R. Figure 5 shows R_{bag} and its congruent match R_{dga}.

Figure 5: *Non-symmetric relative and its congruent match.*

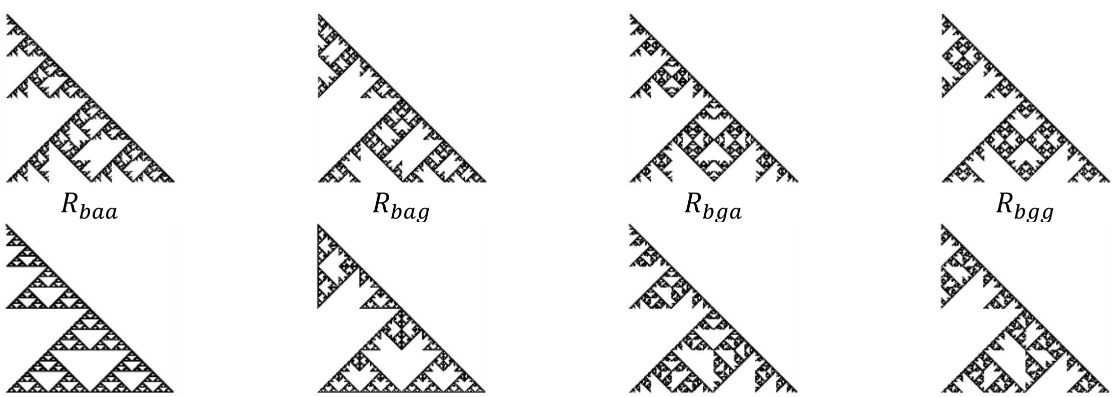
A set X is convex if for any two points p and q in the set, the line segment $\overline{pq}$ joining them is also in the set [5]. The convex hull of a set is the smallest convex set that contains the set. See Figure 6. One can visualize the boundary of a convex hull as being like an elastic band around the points.



convex and not convex　　　　boundary of convex hull

Figure 6: *Convex set and non-convex set; set of points along with boundary of its convex hull.*

Convex hulls of the relatives can be a useful tool to help with the computational topology analysis [12]. The convex hull of a relative can be described by its extreme points (the vertices of the polygon that is the boundary of the convex hull). The extreme points for the gasket are (0,0), (1,0), and (0,1). There are other relatives that have the same convex hull [9,11].



R_{baa}　R_{bag}　R_{bga}　R_{bgg}

R_{faa}　R_{fag}　R_{fga}　R_{fgg}

Figure 7: *Eight relatives with triangle convex hulls.*

Figure 7 displays eight relatives that have the same convex hull: R_{baa}, R_{bag}, R_{bga}, R_{bgg}, R_{faa}, R_{fag}, R_{fga} and R_{fgg}. These 8 relatives are not symmetric. However, they each have a congruent match as described above. These corresponding relatives are: R_{daa}, R_{dga}, R_{dag}, R_{dgg}, R_{eaa}, R_{ega}, R_{eag} and R_{egg}. These 16 relatives are the only non-symmetric relatives that have the same convex hull as the gasket [9,11] and will be referred to as *triangle relatives*.

Square Fractals made with Triangle Relatives

The triangle relatives are not symmetric, but they can be used to make fractals that display different symmetries. This paper focuses on fractals that are considered to be square because their convex hulls have square boundaries. When I first started building fractals with the triangle relatives, I printed off copies of them and played with the paper cut-outs. A directory of images, grouped by specific relative, is available for people to see examples and perhaps create their own [13]. I thought these would be of interest to the Bridges audience. I needed a theme to help decide on which images to share. I chose to show that it is possible to create fractals that demonstrate all non-trivial subgroups of D_4.

For simplicity, the label of a triangle relative R_{xyz} is given by just xyz. The copies may all have the same orientation as the original relative or they may correspond to the congruent match of the relative. The "J" symbol is used to show the orientation of the relative. Table 2 gives an explanation for the figures of the square fractals. Space is limited, so the examples shown use the minimum number of relatives needed for a given subgroup. With two relatives it is possible to build a square fractal that corresponds to subgroups $\{a, c\}$, $\{a, h\}$ or $\{a, g\}$. Figure 8 displays fractals for the subgroups $\{a, c\}$ and $\{a, h\}$. The fractals for $\{a, g\}$ can be obtained by rotating the fractals for $\{a, h\}$ by 90°.

Table 2: *Explanation of figures of square fractals.*

Square to show symmetries and orientation of relative	baa/daa	bag/dga	bga/dag	bgg/dgg
Corresponding subgroup	faa/eaa	fag/ega	fga/eag	fgg/egg

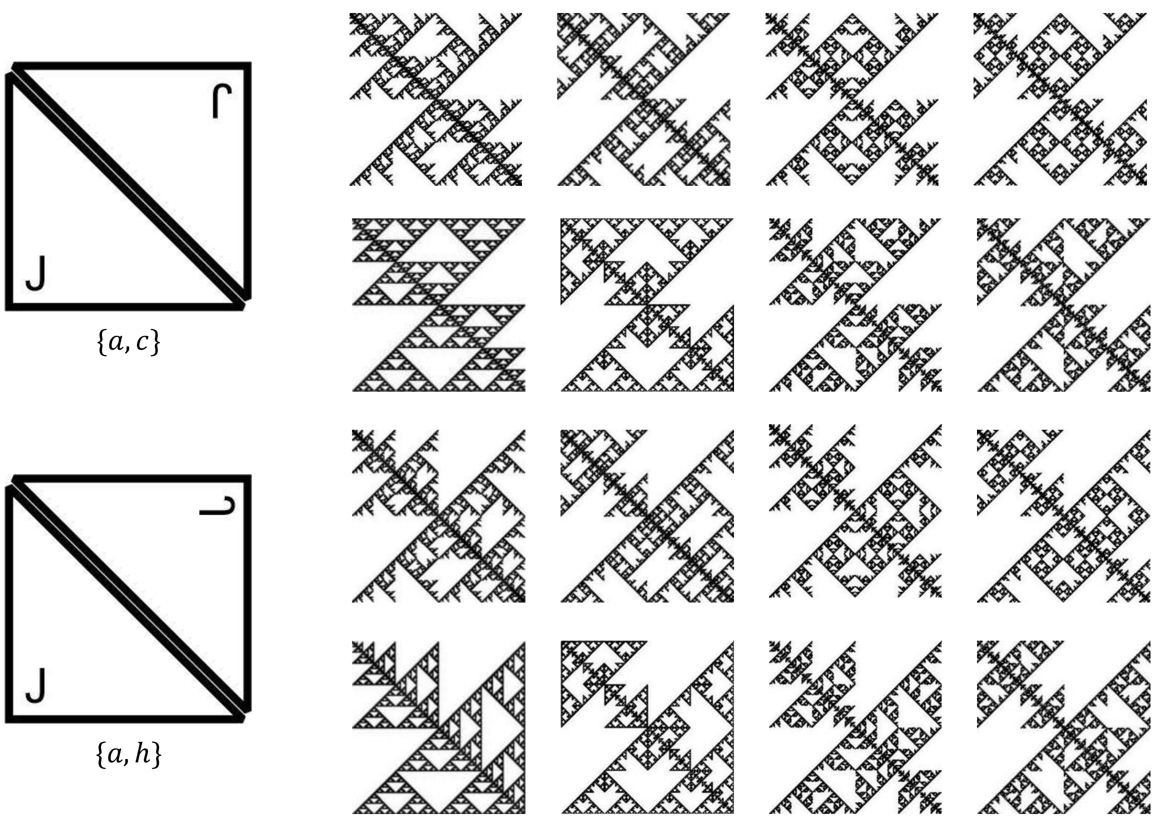


Figure 8: *Square fractals made with two relatives.*

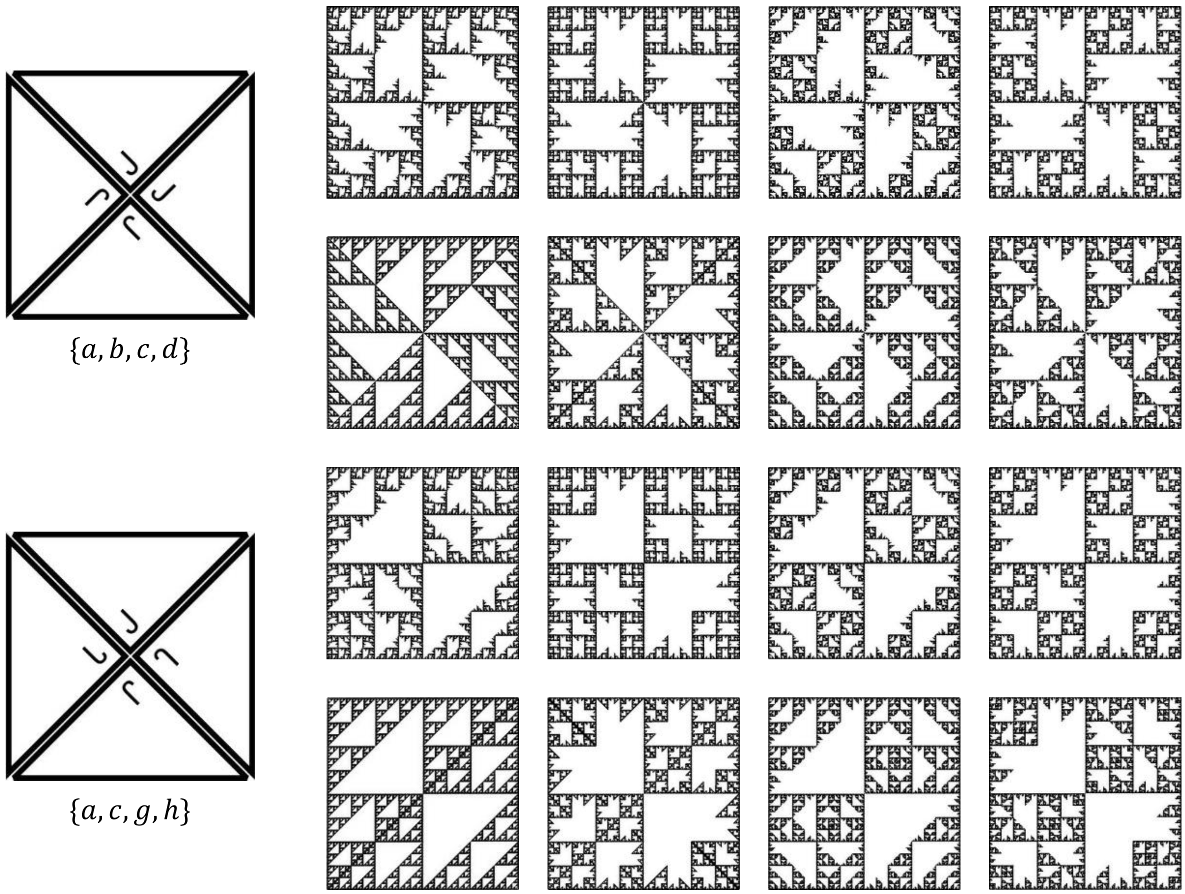


Figure 9: *Square fractals made with four relatives.*

Four relatives can be joined to form a square corresponding to the subgroups $\{a, b, c, d\}$ or $\{a, c, g, h\}$, see Figure 9. Eight triangle relatives can be put together in many different ways. Figure 10 shows square fractals corresponding to the subgroup $\{a, f\}$. These could be rotated by $90°$ to get fractals corresponding to the subgroup $\{a, e\}$.

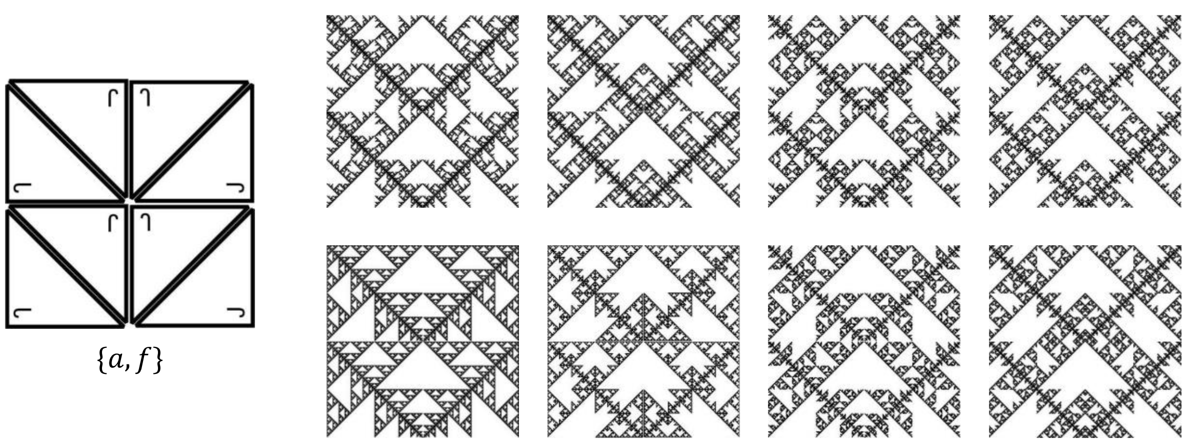
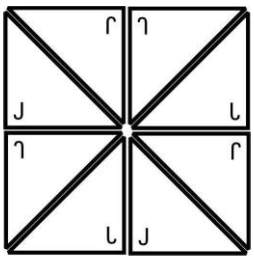


Figure 10: *Square fractals, made with eight relatives, corresponding to* $\{a, f\}$.



$\{a, c, e, f\}$

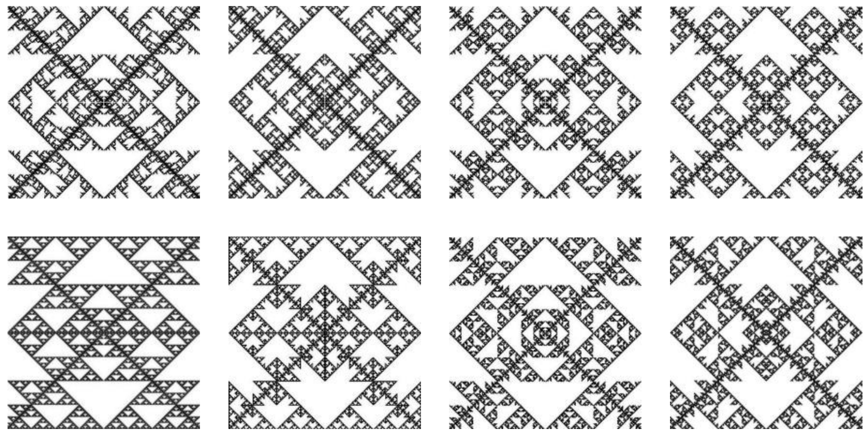
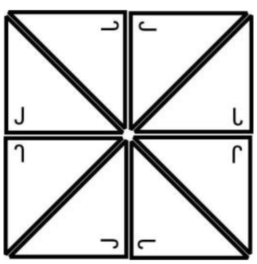
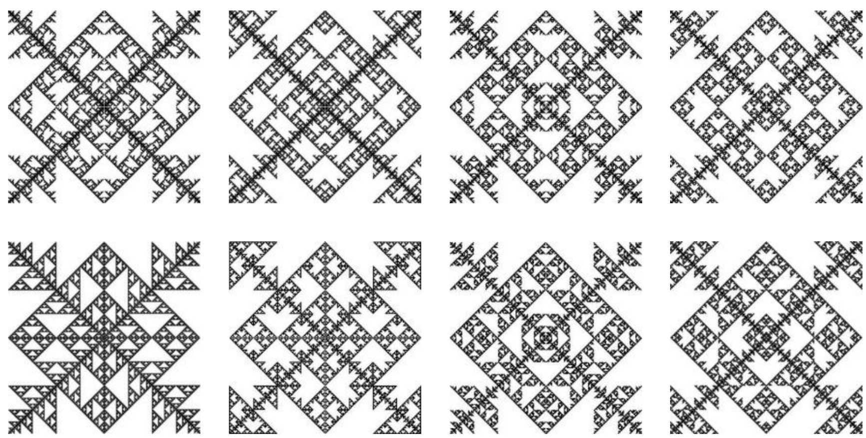


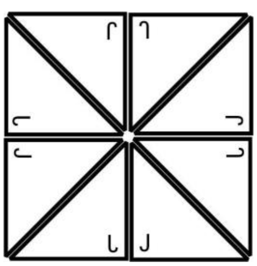
Figure 11: *Square fractals, made with eight relatives, corresponding to* $\{a, c, e, f\}$.

Figure 11 shows square fractals that correspond to $\{a, c, e, f\}$. Figure 12 presents two different formations that yield fractals with all symmetries of the square.



$\{a, b, c, d, e, f, g, h\}$





$\{a, b, c, d, e, f, g, h\}$

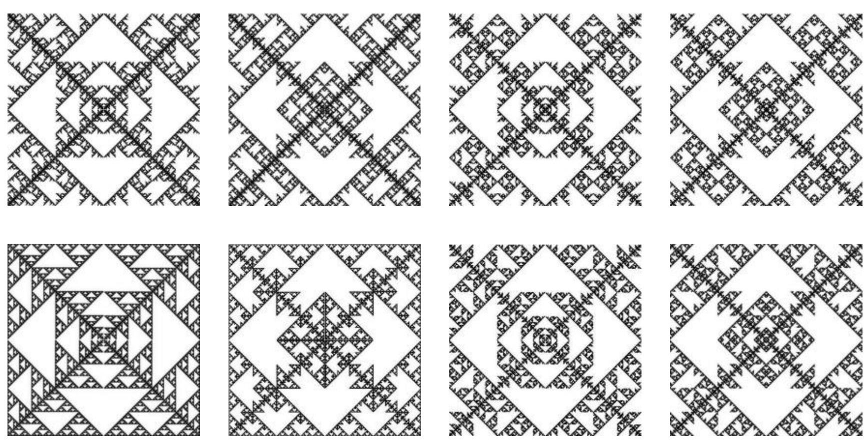


Figure 12: *Square fractals, made with eight relatives, corresponding to* D_4.

Summary and Conclusions

This paper has presented a way to use triangle relatives to form square fractals that help to visualize the different subgroups of the symmetries of the squares. I find it interesting to see how different themes emerge depending on how the relatives are joined together. Isosceles right triangles can be used to build many other interesting and beautiful figures. For example, two spirals are shown in Figure 13. The directory includes other figures such as frieze patterns and shapes with convex hulls different from the triangle and square [13]. It would also be interesting to look at triangle relatives that correspond to the equilateral triangle version of the gasket; they could be used to form fractals whose convex hulls have hexagonal boundaries.

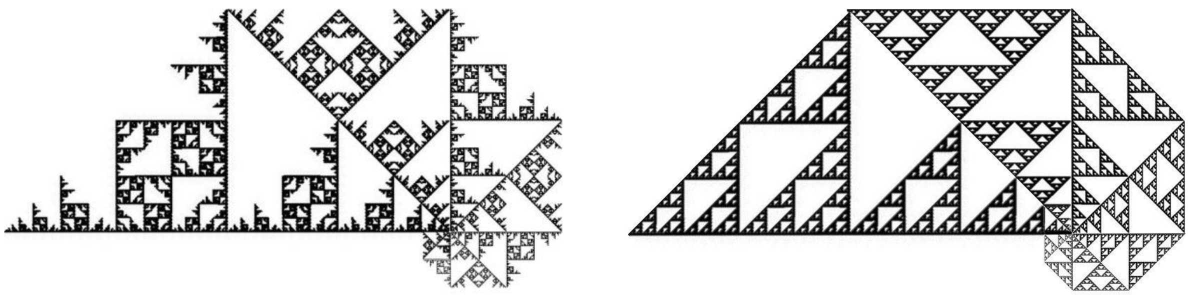


Figure 13: *Spirals made from triangle relatives* R_{bga} *and* R_{faa}.

Acknowledgements

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