Hyperbolic Flyby into a Schwarzian Expanse

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Abstract

We present our continued explorations in decorating the Schwarz P surface with Escher patterns. Furthermore, we present a viewpoint from inside this triply periodic surface through a flyby animation. In addition, we introduce a near-miss Johnson solid that can partially serve as a lightweight substitute for the Schwarz P surface.

Introduction

In 2022, we investigated the use of the Schwarz P surface for tiling periodic patterns [3]. This paper is a continuation of this work. Our results are shown in Figures 1 and 2. The flyby animation [4] that we created to illustrate the structures in this paper gives further insight into the beauty of this surface.

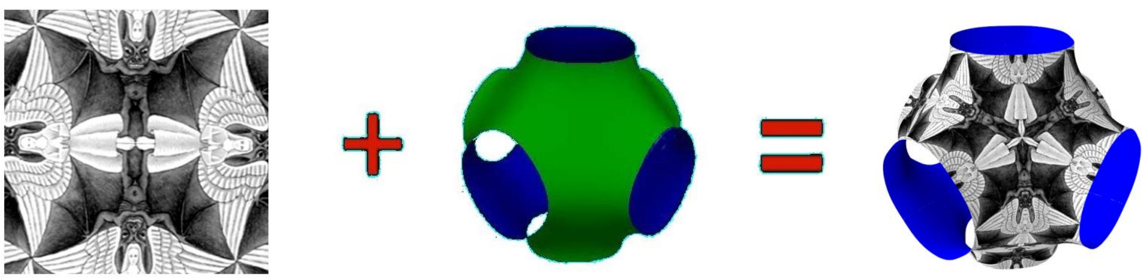


Figure 1: *Escher's devilish texture on the Schwarz P surface.*

A triply periodic surface is a surface that repeats indefinitely in three independent directions of Euclidean space. The Schwarz P surface is an example of a triply periodic surface. It was originally studied by Hermann Schwarz in the 1860s within the context of minimal surfaces. Figure 1 shows a unit cell of the Schwarz P surface overlaid with an Escher texture.

In our Bridges 2022 paper [3], we primarily focused on one cell of the Schwarz P surface and did not really delve into its triply periodic nature. At most, we only dealt with a group of 3×3×3 cells from the surface, which is a pittance compared to its true infinite expanse. It goes without saying that we fell short in showing the Schwarz P in its full glory. In this paper, we will redress this shortcoming. Moreover, we embellish the surface by decorating it with infinite M.C. Escher patterns.

Figure 2: *Our flyby animation reel with a viewpoint from inside the Schwarz P surface.*

Escher himself was interested in triply periodic patterns. This is evidenced by two of his lithographs: *Cubic Space Division* (1952) and *Depth* (1955), shown in the left side of Figure 3. These artworks show that Escher was a true aficionado of infinite space. In addition, Escher was also interested in regular divisions of the plane in the form of periodic tessellations. We share in Escher's affinity with infinity. In this paper, we will combine two of Escher's passions into a single coherent art piece. We do this by overlaying his regular tessellations on the triply periodic Schwarz P surface. Furthermore, in order to highlight the unbounded periodicity of this surface, we provide a viewpoint from inside it through a flyby animation. This is the main novelty of this paper.

Figure 3: *Our flyby animation was inspired by Escher's artwork.*

Schwarzian Expanse

Triply periodic polyhedra [2] form an interesting faceted subset of triply periodic surfaces. These shapes consist of adjoining polygons that repeat indefinitely in 3 independent directions of Euclidean space. For this section, we are primarily interested in two specific examples of triply periodic polyhedra: the *mucube* [1][5] and its dual, the *muoctahedron* [1]. These two shapes are topologically equivalent to the Schwarz P surface and served as an important framework to the surface tessellations done in this paper. A Polydron™ model of an extended mucube cell is shown in the left diagram of Figure 4. Similarly, a Polydron™ model of the muoctahedron cell is shown in the right diagram of Figure 4. Both models are flanked by a unit cell of the Schwarz P surface at the center of Figure 4.

In 2012, Dunham [2] decorated the mucube and the muoctahedron with artful tessellations by M.C. Escher. Dunham did this after realizing that tiling patterns on the Poincaré disk model of the hyperbolic plane can be transferred to triply periodic polyhedra. Ten years later [3], we analogously transferred tiling patterns from the mucube to the Schwarz P surface. We continue this work here by transferring tiling patterns from the muoctahedron to the Schwarz P surface. This is one of the major novelties achieved in this paper.

The truncated octahedron is a well-known Archimedean solid consisting of 8 regular hexagons and 6 squares. The unit cell of the muoctahedron comes from removing all the squares of the truncated octahedron and leaving them as six holes. These square holes are then used to connect with other muoctahedron cells in order to produce a triply periodic polyhedron. A paper model of several muoctahedron cells arranged in a 2×2×2 network is shown at the center of Figure 5. At the bottom left corner of the same figure, we show a tiling pattern on the Poincaré disk model of the hyperbolic plane. This pattern can be transferred to the muoctahedron and the Schwarz P surface at the right of the figure.

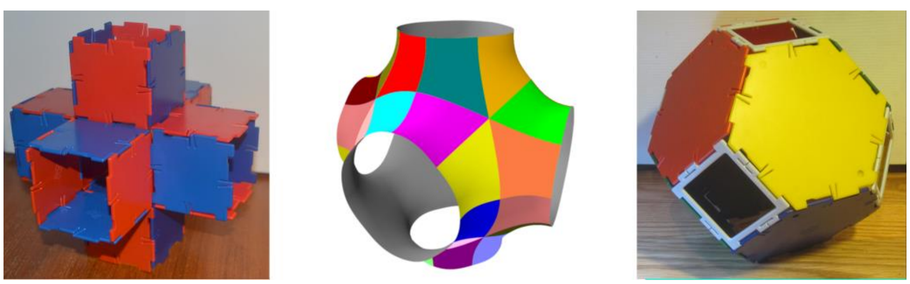


Figure 4: *Schwarz P unit cell (center) flanked by corresponding mucube & muoctahedron.*

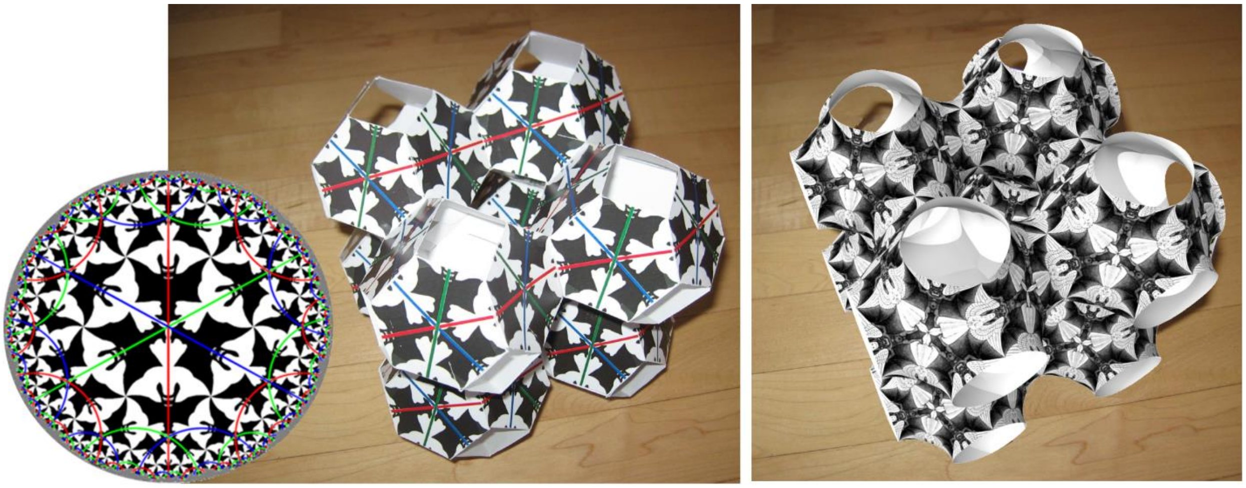


Figure 5: *A side-by-side comparison of 2×2×2 cells of the muoctahedron and the Schwarz P surface.*

In order to visualize the Schwarz P surface with texture, we had to represent it in the computer. We did this by approximating the Schwarz P surface using Bézier surfaces. Our geometric model for the unit cell of the Schwarz P surface consists of 48 bicubic Bézier patches. We utilized the highly symmetric nature of the Schwarz P surface in order to use a single fundamental patch that is replicated and transformed 48 times to span the whole unit cell of the surface. These affine transformations include reflection, translation, and rotation in 3D space. These patches are shown in different colors at the center of Figure 4. There are more details about this in our previous paper [3].

A Lightweight Polyhedral Alternative

In this section, we will present a faceted model that can serve as lightweight substitute for the Schwarz P surface. This faceted model requires much less computer memory than smooth surfaces. During the course of the development of our flyby animation, we found that our Bézier patch model has a prohibitive memory footprint that would hog resources on the computer. This was further exacerbated by the fact that we need to display thousands of these patches during a flyby. Consequently, we created a faceted model with moderate memory requirements and yet still has a reasonable resemblance to the Schwarz P surface. This allowed us to model the Schwarz P surface at a lower level of detail when needed. In computer graphics, *level of detail* or *LOD* for short, is an important concept in which far away objects are represented at a lower amount of geometric complexity than the actual underlying model. LOD was an indispensable optimization tool that we used in the development of our flyby animation.

Our lightweight Schwarzian substitute is shown in Figure 6. This includes a front view on the left followed by two perspective views. It is a faceted model consisting of 84 squares and 24 irregular pentagons. Note that this shape has octahedral symmetry. This means that its front view is identical to its back view. The same holds for its top view, bottom view, as well as left and right views.

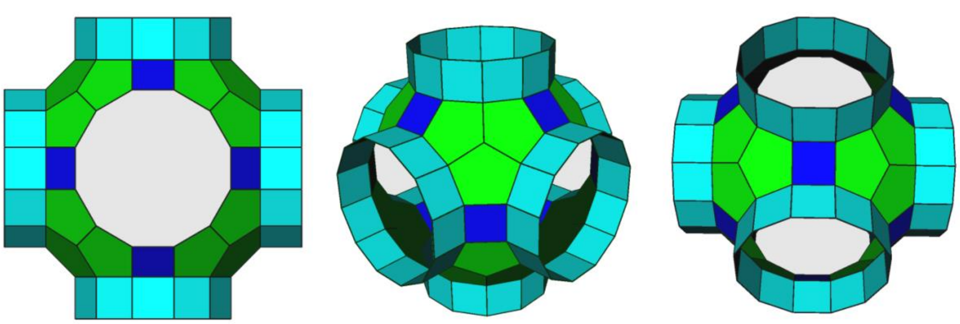


Figure 6: *Our lightweight polyhedral alternative to the Schwarz P surface.*

Much like the unit cells for the mucube and the muoctahedron, our lightweight Schwarzian substitute is not a closed polyhedron. It has 6 open orifices that have a contour in the shape of a regular dodecagon. These orifices are analogous to the square holes in the mucube and the muoctahedron, which are used to connect with other cells to form a triply periodic shape that extends throughout space. The mucube, the muoctahedron, and our lightweight Schwarzian substitute belong to a family of infinite polyhedra [1] known as skew apeirohedra. Note that '*apeiro*-' is a prefix from ancient Greek meaning infinite.

When Hermann Schwarz originally studied his namesake primitive surface, he observed that the orifice is almost circular in cross-section. He mentioned that the 2D shape encompassing the orifice had radial errors measuring less than 0.4% from an actual circle [5]. For our purpose of approximating the Schwarz P surface as an open polyhedron, we made the simplifying assumption that the orifice is completely circular. This is the reason why we chose to have a dodecagonal outline for the orifices in our model. We wanted our lightweight model to have a small number of polygonal facets yet still maintain a fairly circular orifice. The regular dodecagon offers a good compromise between polygonal simplicity and circular verisimilitude.

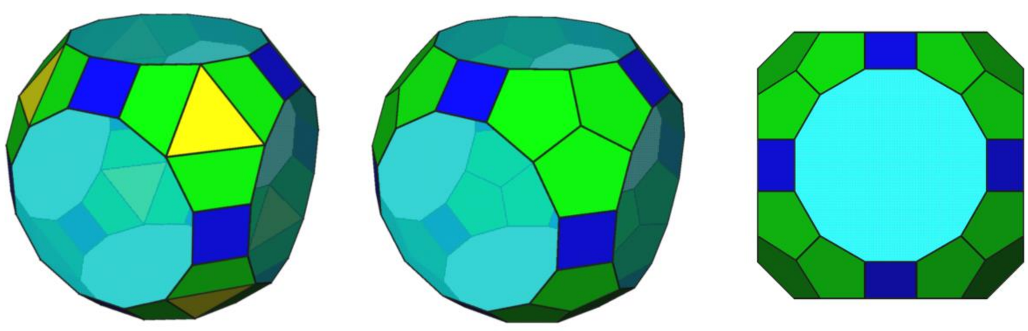
We have included data files for our lightweight Schwarzian substitute as a supplement to this paper. Our models come in the OFF file format which can be viewed in 3D on a computer using the *Antiview* software program. *Antiview* comes as part of the *Antiprism* software distribution [8], which is a free and open-source software package for examining and experimenting with polyhedra.

The Schwarz P surface separates 3D space into two congruent labyrinths [5]; and has the symmetry group $Im\overline{3}m$, also known as space group #229. Unfortunately, our lightweight Schwarzian substitute does not separate 3D space into two congruent labyrinths. Consequently, it has less symmetry, namely space group #221.

Symmetrohedra

In 2001, Craig Kaplan and George Hart introduced an infinite family of highly symmetric polyhedra [7]. We used a specific instance of their symmetrohedra as the basis for our lightweight Schwarzian substitute. This is shown in Figure 7(a). In the Kaplan-Hart notation [7], this polyhedron is known as the O(3,*,2,e) symmetrohedron. It has octahedral symmetry and appears almost cubiform in structure. It consists of 6 regular dodecagons, 12 squares, 8 equilateral triangles, and 24 isosceles trapezoids. Each of the dodecagons corresponds to a face on the cube. Similarly, each of the triangles corresponds to a corner vertex of the cube. These corner triangles are surrounded by 3 trapezoids.

We performed a slight modification of the O(3,*,2,e) symmetrohedron in order to get the core of our lightweight Schwarzian substitute. This convex core is no longer a symmetrohedron and is shown Figure 7(b). Our modification consists of replacing each corner triangle and its surrounding trapezoids with three pentagons. These replacement pentagons are all congruent with each other but are not regular. We show a front view of our modification in Figure 7(c).



(a) (b) (c)

Figure 7: *Our dodecagonal near-miss Johnson solid shown in (b) and (c) is a modification of the O(3,*,2,e) symmetrohedron shown in (a).*

Near-Miss Johnson Solids

A *Johnson solid* is a strictly convex polyhedron with each face being a regular polygon. In 1966, Norman Johnson published a list of 92 such polyhedra. His list excludes uniform polyhedra such as the Platonic solids and the Archimedean solids. His list also excludes the infinite family of uniform prisms and antiprisms. A few years later, Victor Zalgaller proved that Johnson's list was complete. This means that there are exactly 92 Johnson solids. One nice property of a Johnson solid is that all of its edges have the same length, which we can assume to be 1 for our discussion here.

A *Near-Miss Johnson solid* [9] is a strictly convex polyhedron with each face close to being a regular polygon. Our modification to the O(3,*,2,e) symmetrohedron is a near-miss Johnson solid. All of its polygons are regular except for 24 pentagons which are nearly regular but not quite. These corner pentagons are very close to being equilateral. The left diagram of Figure 8 shows the measurements of our corner pentagon. Two of the sides have a length of 0.9922 instead of 1. Also, all five interior angles are reasonably close to the 108^{o} required for regular pentagons.

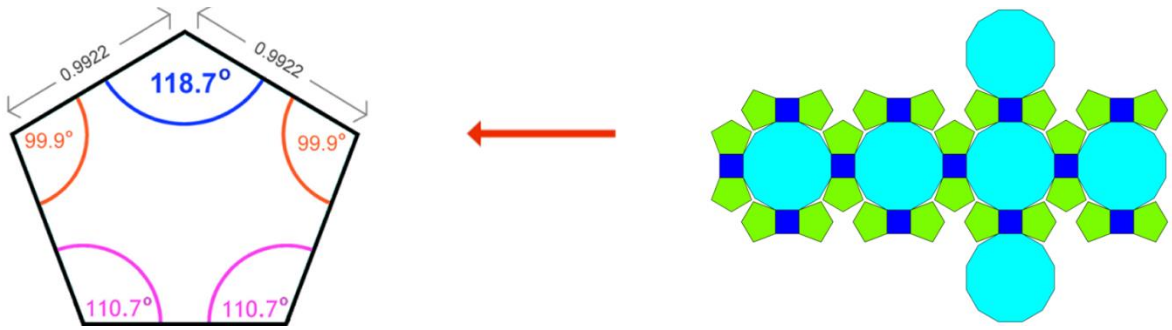
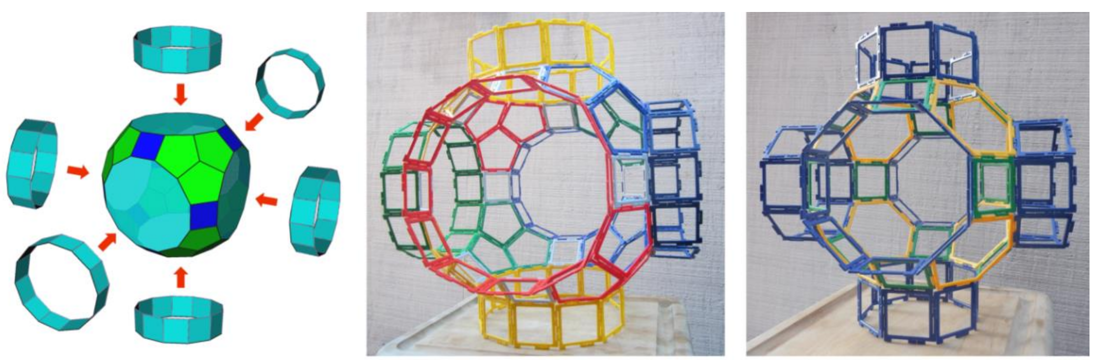


Figure 8: *Polyhedral net for our dodecagonal near-miss Johnson solid.*

We have included a polyhedral net for our dodecagonal near-miss Johnson solid in Figure 8. This polyhedron serves as the convex core for our faceted alternative to the Schwarz P surface. Afterwards, we perform a prismatic extension on this convex core to bring forth our lightweight Schwarzian substitute. An exploded view of this extension process is shown in Figure 9(a).

We have built a plastic model of our lightweight Schwarzian substitute using Polydron™ frameworks. This is shown in Figure 9(b). Even though there is structural stress on all the irregular pentagons in the model, the whole structure is reasonably rigid. This plastic model also shows the main reason why we chose our lightweight Schwarzian substitute over other comparable polyhedra arising out of the O(3,*,2,e) symmetrohedron. Basically, we want to be able to build it out of Polydron™ parts, which makes it tangible. Also, we have great interest in near-miss Johnson solids.



(a) (b) (c)

Figure 9: *(a) Our lightweight Schwarzian substitute consists of a convex core with 6 extensions.*
(b) Polydron™ plastic model measuring 20 inches across on all 3 major axes.
(c) Polydron™ plastic model of a prismatic extension of the truncated cuboctahedron.

Our final diagram for this paper shows the feasibility of our lightweight Schwarzian substitute when compared to the real thing. We show a side-by-side comparison of multiple cells with texture in Figure 10. Although the triply periodic polyhedral model on the left is coarse and faceted, it still has a reasonable semblance to the Schwarz P surface on the right.

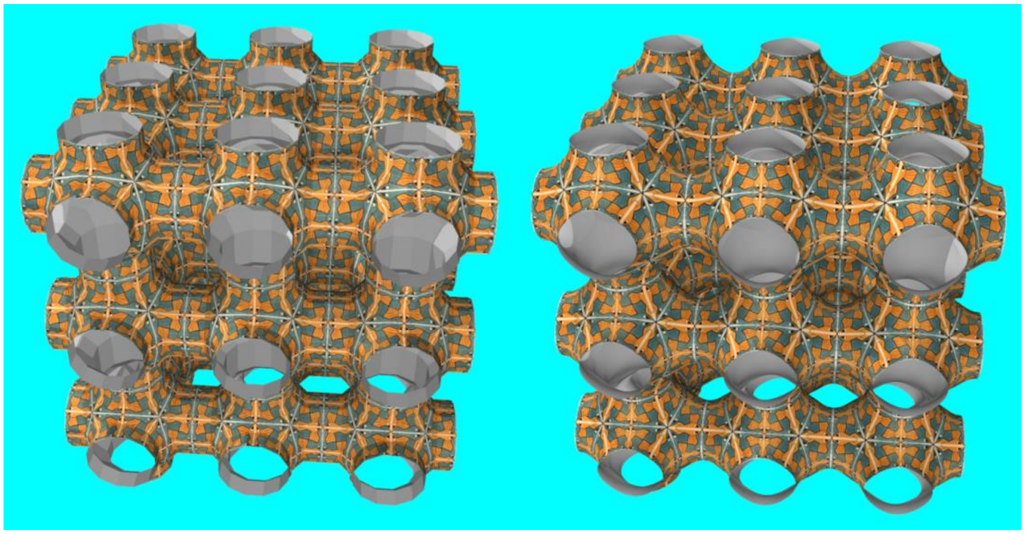


Figure 10: *A textured comparison of our faceted alternative to the Schwarz P surface.*

Summary and Future Work

We produced a flyby animation through the triply periodic Schwarz P surface decorated with several Escher-inspired patterns. We also proposed a lightweight faceted alternative to the Schwarz P surface, which was used for level of detail optimization in our flyby. For future work, we wish to consider the prismatic extension of the truncated cuboctahedron as another type of LOD. A Polydron™ model of this is shown in Figure 9(c). This polyhedron has an octagonal orifice which is less circular than the regular dodecagon, but it has $Im\bar{3}m$ symmetry which is very desirable. In other avenues of future work, we would like to do a flyby inside triply periodic surfaces such as the gyroid and the Schwarz D surface.

Acknowledgements

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