Design Equations for Grid-Based Origami Tessellations

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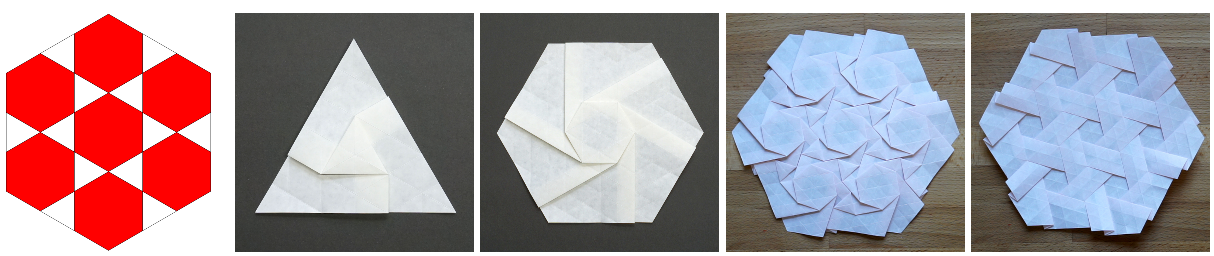
Abstract

Twist structures in origami tessellations can be discretized and placed on a number line with a fixed transformation between each twist. These number lines represent not only potential for infinite expansion of twists, but also the connection between twists on opposite sides of the paper. When extended symmetries are applied to tessellation tilings these number lines of twists provide relative positions that enable the construction of design equations representing the pleat shifts needed to produce the given symmetry. These extended symmetries enable the construction of ever-more elaborate and numerous origami tessellation designs while using only the simplest twists and folding methods.

Introduction

Origami tessellations are repeating geometric patterns folded from a single sheet of paper [5]. I will be discussing specifically the subtype of origami tessellations that are folded on a regular background grid of squares or equilateral triangles and composed of flat-folded twists connected by pleats where the parallel mountain and valley folds are one grid spacing apart. A *twist* is an interaction of pleats such that their mountain and valley folds alternate around some center. Generally, this center will be a polygon with either all mountain or all valley folds. These arrangements of pleats are often, but not necessarily, rotationally symmetric and all twists have a rotation direction, either left- or right-handed [3]. Twist-based origami tessellations are typically folded from the center out in concentric rings with one twist added at a time.



(a) (b) (c) (d) (e)

Figure 1: *From tiling to tessellation (a)* $(3.6)^2$ *vertex configuration tiling, (b) Closed (back outlines a point) triangle twist in a triangle tile, (c) Open (back outlines an area) hexagon twist in a hexagon tile, (d) tessellation folded with the selected tiling and twists, as seen from the front, (e) same tessellation as seen from the back. The area (or point) outlined on the back of a twist corresponds to the area outlined in the center of the pleat arrangement of the twist.*

We can consider the arrangement of pleats separately from how any particular arrangement of pleats folds flat by considering a *tile* with edges perpendicular to each pleat in a given arrangement. In Figure 1 I show the relationship between a tile and a twist for the triangle and hexagon cases. The specific arrangement of pleats relative to the center of the tile determines whether the twist has an open or closed hole on the back of the paper. All twists with a particular number and spacing of pleats fit in the same tile, so all twists with three pleats spaced evenly apart from each other fit into an equilateral triangle tile, and all twists with four pleats spaced evenly apart from each other fit into a square tile. In the terminology of [3], I am using centered tiles with pleats that cross the tile boundary perpendicularly.

A *tiling* is an arrangement of tiles that indicates the structure of an origami tessellation [3]. Since I am only discussing twist-based tessellations, I will only use 2-colorable tilings since all right-handed twists need left-handed neighbors (and vice versa) in order to have the same pleat connecting two twists. The discussion will also be limited to 1-uniform tilings with regular polygon tiles for brevity, which will be referred to by their vertex configurations. Tilings are an abstraction of the complexity of a crease pattern and cannot be directly overlaid onto a crease pattern in most cases using twist-centered tiles with edge-perpendicular pleats. However, these conditions do allow us to make statements about the symmetry observed in the crease pattern and folded forms by identifying symmetry positions on the tiling level, which allows us to identify distinct positions in the tiling that can be filled by different twists. In Figure 1, all hexagon tile centers were assigned positions of 6-fold rotational symmetry and all triangle tile centers were assigned positions of 3-fold rotational symmetry. Given these constraints, each hexagon tile was filled with the same (right-handed open front) hexagon twist and each triangle tile was filled with the same (left-handed closed front) triangle twist. Either the hexagon or the triangle could be swapped out for any other twist matching the same tile, as long as all of the twists in tiles of that shape are the same.

Number Lines of Twists

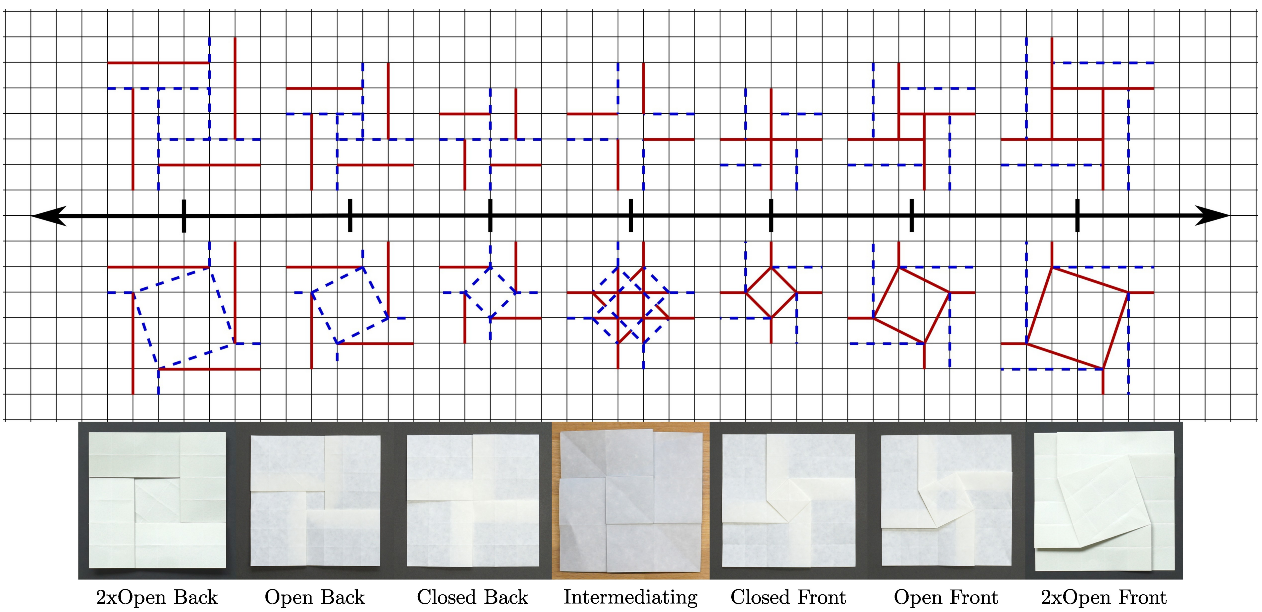
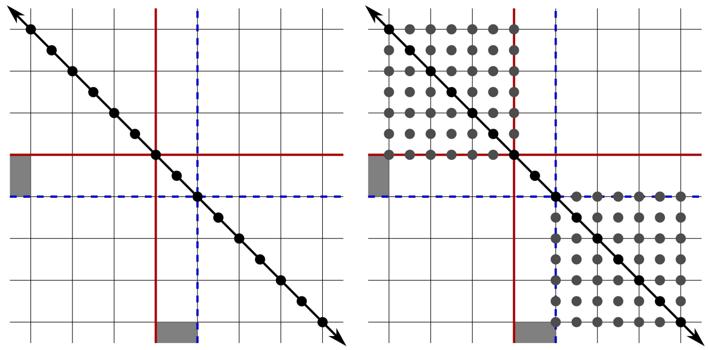


Figure 2: *Number line of square twists from the tiling perspective with the pleat arrangement, flat-foldable crease pattern, folded photo, and name of the seven central twists*

So, let's discuss the variability available for pleat arrangements in a given tile—the square tile. As you can see in Figure 2, there are infinitely many ways to arrange four pleats in a 4-fold rotationally symmetric fashion on a square grid. These arrangements vary in their perpendicular distance between pleats on opposite sides of the twist, and in Figure 3 I show that their centers form an evenly spaced line relative to two fixed pleats. The two fixed pleats are used for referencing the center of these twists because this situation is ubiquitous in the process of folding these origami tessellations twist by twist. Given some twists already on the paper, you need to decide which square twist to fold at a particular intersection of pleats. In this sense, the number line of square twists is a list of affordances available for folding at a perpendicular pleat intersection.



(a)

(b)

Figure 3: *Number line of square twists from the folding perspective with (a) centers of 4-fold rotational symmetry (square line) and (b) centers of 2-fold rotational symmetry that are flat-foldable, given two fixed pleats. The gray boxes at the edge mark the fixed pleats, so these left and bottom pleats match all the left and bottom pleats in Figure 2 and the more-open-front direction is up and left.*

My claim is that these arrangements of pleats form a number line on which we can count distances, and yet I am not numbering this number line. Attempts have been made to develop a comprehensive notation for pleat arrangements [1][2], and there is an opportunity for future work in this area, but I have found that these systems lock you into either the tiling (Figure 2, center-based) or the folding (Figure 3(a), pleat interaction-based) perspective and that the ability to use both perspectives is better than either alone.

My second reason for declining to number my number lines of twists is the varying number of *intermediating* twists—none, one, two, or three—in different tile shapes. These twists occupy position(s) between the closed twist on the front of the paper (centered at the intersection of mountain folds as in Figure 3) and the closed twist on the back (centered at the intersection of valley folds) and require additional precreasing to collapse cleanly. In the square and equilateral triangle cases (where the tiles are regular polygons), these twists are still flat-foldable, but with additional creases needed as seen in Figure 2. While the square case would work well with positive and negative integers and zero assigned to the intermediating twist, this system would not work for regular hexagonal or triangular arrangements of pleats, so I will rely on distance measures with a value of 1 between consecutive arrangements of pleats with the more-open-front direction defined as positive.

Hybrid Square Twists

Since the only requirement for an arrangement of pleats to fit in a given tile is that the pleats cross the tile boundaries perpendicularly, arrangements of pleats that do not match the full rotational symmetry available in a given tile will also fit in that tile. In other words, a rectangular arrangement of pleats fits in a square tile just as well as a square arrangement of pleats. However, not all of these rectangular arrangements of pleats are flat-foldable. Only the arrangements that function as combinations of two regular square twists on the same side of the paper fold flat without self-intersection. A *hybrid square twist* centered on a given position in Figure 3(b) combines the vertically-nearest twist centered on the square line in Figure 3(a) with the horizontally-nearest twist centered on the square line to create a rectangular arrangement of pleats that is folded flat through a combination of the features of the two parent twists, as described in Figure 4(a). These twists can be categorized by three variables: the *gap* between the vertical and horizontal spacings of pleats (equivalent to the distance from the square line), the *size* of the smaller (more central) twist in the hybrid, and the *orientation* of the larger distance between pleats in the twist being either horizontal or vertical. If

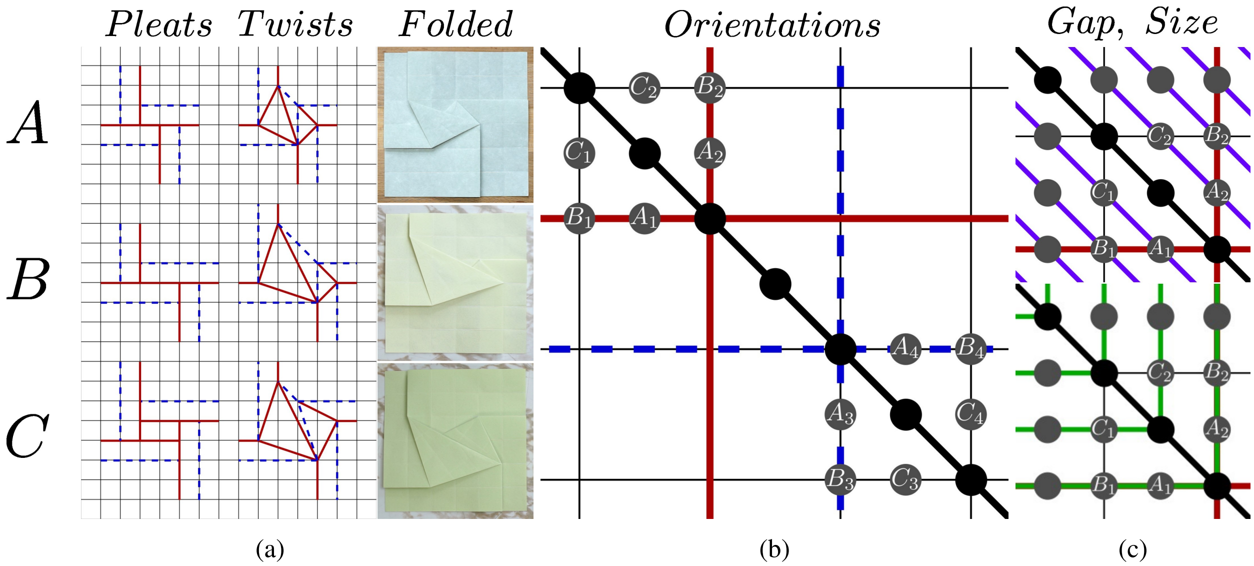


Figure 4: *Hybrid square twists (a) Closed-Open, Closed-2xOpen, and Open-2xOpen hybrid square twists as pleat arrangements, flat-foldable crease patterns, and folded forms, (b) orientations of hybrid square twists, which have an additional horizontal/vertical component that is not seen in regular square twists, (c) illustrations of gap and size of hybrid twists in orientations 1 and 2, which are the distance from the central square line and the identity of the smallest square in the hybrid, respectively. Subfigures (b) and (c) have the same structure as Figures 3(a) and (b).*

we consider the twist to be the same regardless of the side of the paper that we view it on, there are four orientations for each hybrid combination of two regular square twists. The crease pattern and folded forms of hybrid square twists are not rotationally symmetric, but since the pleat arrangement is 2-fold rotationally symmetric, there are two options for how to fold each hybrid square twist in a given orientation.

When considering the broader symmetry of an origami tessellation that uses hybrid twists (such as when assigning symmetry positions on a tiling), I categorize these twists by the symmetry of their pleat arrangements instead of their folded forms since symmetry relationships are determined by the locations of the centers of twists as determined by their pleats. Twists that fit in hexagon and rhombus tiles may also be hybridized in a similar manner, with the types of hybridization corresponding to the ways the tile can be dissected with non-intersecting cuts made only between vertices of the original tile (for squares, we made a single cut between opposite vertices). Thus, a triangle cannot be hybridized, and a hybridized hexagon can have 3-fold, 2-fold, or no rotational symmetry depending on the dissection, as shown in Figure 5. The rotationally-symmetric hybrid hexagon twists will also have rotationally symmetric folded forms, unlike the hybrid square twists.

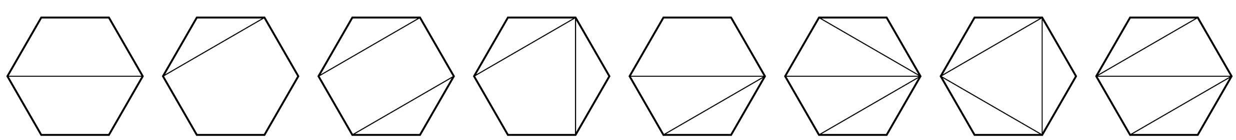
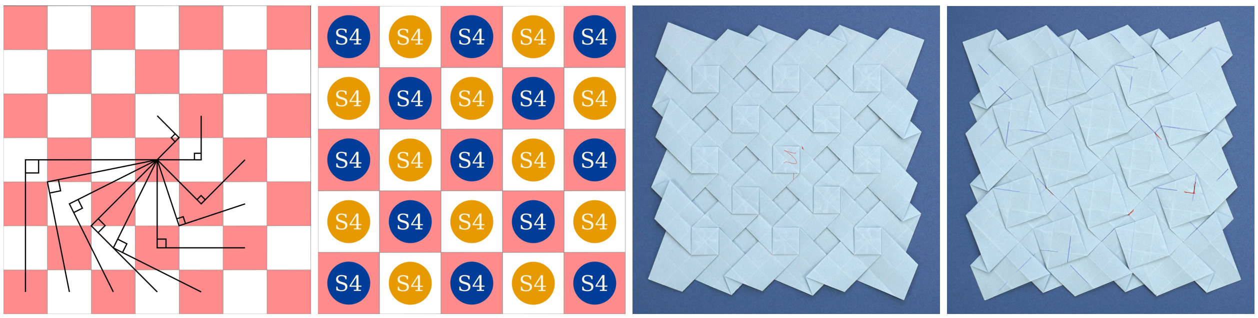


Figure 5: *Dissections of a hexagon tile that can be used for hybridizing hexagon twists. The third and seventh dissections may be centered on 2-fold and 3-fold rotationally symmetric points, respectively, if only two parent twists are used in the hybrid twist. The sixth and eighth dissections must use at least three parent twists.*

4^4 Tiling Symmetry Extensions

Now that we've seen the full variety of twists available to fill a tile (with a fixed pleat depth and single pleat per tile edge), it's time to return to how symmetries can be applied to tilings. The simplest option is to assign the maximum symmetry possible to each tile center (and thereby to each twist center), with any additional symmetry points (needed to comply with the relevant planar symmetry group) assigned to corners of tiles, which translate to spaces between twists in the folded origami tessellation. This is the *maximum symmetry density* of the tiling, and the 4^4 tiling example is shown with its maximum density of 442 symmetry (Conway notation) in Figure 6(b-d) and nine additional options for 442 symmetry placement in Figure 6(a).

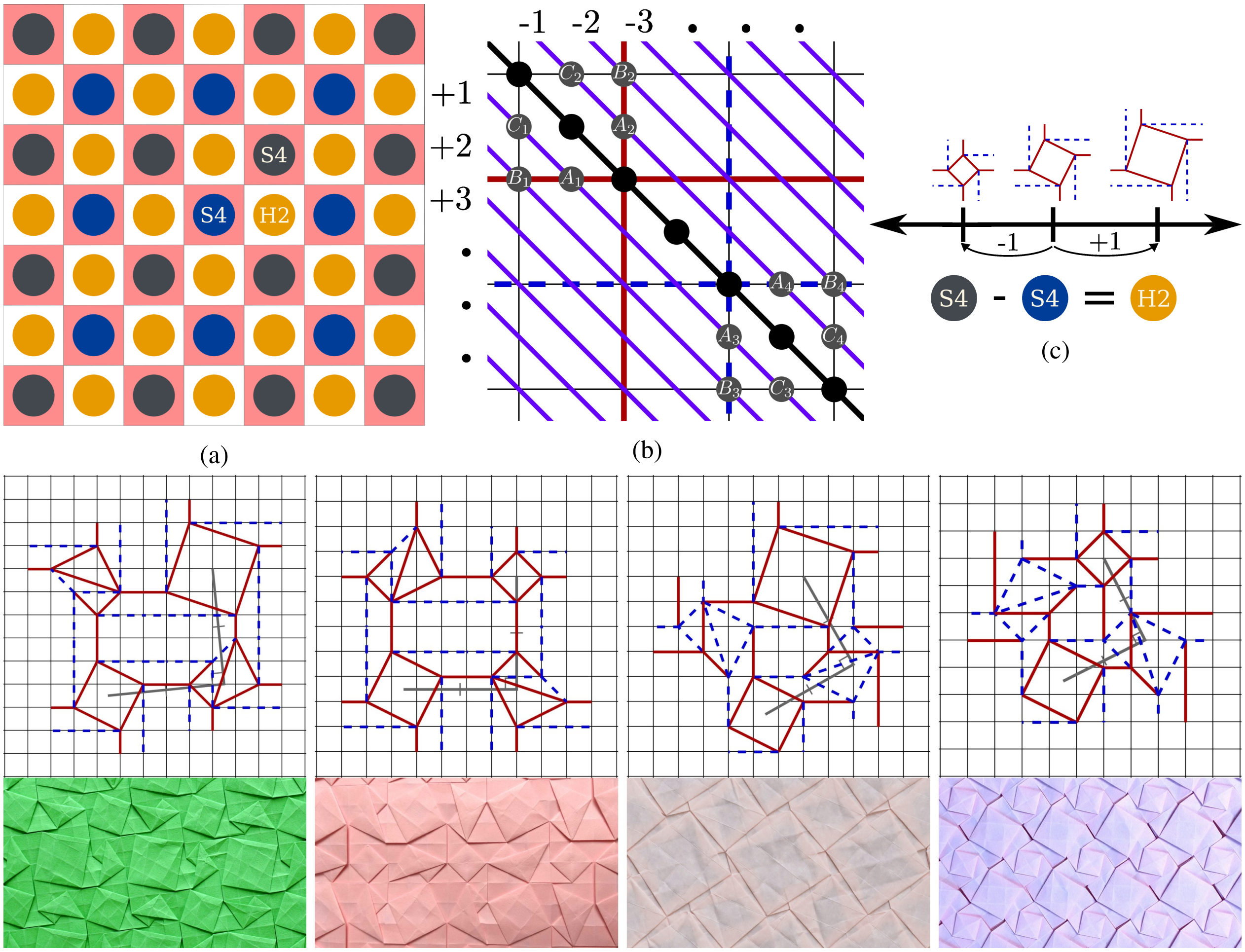


(a) (b) (c) (d)

Figure 6: *Symmetry density and alternating twists on a 4^4 tiling: (a) symmetry density options shown as right angles on positions of 2-fold rotational symmetry with line segments ending on distinct positions of 4-fold rotational symmetry (b) a 4^4 tiling with two distinct positions of 4-fold rotational symmetry in square twist centers and (c) an alternating pattern of open and closed square twists on opposite sides of the paper, with positions of 4-fold rotational symmetry in the center of every twist and positions of 2-fold rotational symmetry between closest pairs of the same twists, (d) the back of this tessellation.*

The assignment of symmetry on a tiling determines the number of variables that can be changed in the origami tessellations derived from that tiling-symmetry combination. With the maximum symmetry density on a 4^4 tiling as shown in Figure 6(a), there are three variables: the contents of the blue-dot tiles, the contents of the gold-dot tiles, and the number of grid spacings between those two twists. Both tiles must be filled by regular square twists and so if you consider only the common options of closed and open square twists then you have only eight resulting tessellations before applying the spacing variable—Closed/Closed/Same Side (CCS), CC/Different Side (CCD), C/Open/S (COS), COD, OCS, OCD, OOS, OOD. Additionally, the COS/OCS and COD/OCD pairs are the same up to reflection (different assignment of right- and left-handed twists), so we effectively have only six options for origami tessellations with common twists before applying the spacing variable.

In the search for additional origami tessellations to fold with only commonly-used twists on the 4^4 tiling, there are several ways to increase the options available. One way is to include hybrid twists in the options available for alternation, which reduces the the center of each twist to 2-fold rotational symmetry. Another way is to selectively add space between pairs of rows and/or columns of twists in one of the alternating options—since all pleats cross tile boundaries perpendicularly, a straight line drawn on tile boundaries in a tiling is only crossed by pleats perpendicular to that line and space can be freely added [4]. I call these lines *tiling breaks*. A third way is to mirror rectangular collections of twists. A fourth way is to assign open or closed properties to rows and columns of twists all on the same side, with hybrid twists at open-closed intersections. Last but certainly not least, we can assign a sparser symmetry density to the tiling to increase the number of distinct positions of twists and thereby increase the choices available for tessellations.



(a) (b) (c)

(d)

Figure 7: *First 442 symmetry extension on the 4^4 tiling (a) three distinct positions in the tiling, two with 4-fold rotational symmetry and filled with regular square twists and one with 2-fold rotational symmetry and filled with a hybrid square twist (note the location of H2 for orientation purposes), (b) assignment of positive values of gap for hybrid orientations 1 and 3 and negative values for orientations 2 and 4, (c) if the blue-to-gray position square twist shift is to the right, then the tessellation must have used a positive-value hybrid gap orientation (1 or 3) and vice versa for a shift to the left, (d) examples of all four orientations of a Closed-Open hybrid square twist as applied to an Open Front square twist in the blue S4 position in both crease pattern and folded forms. The third folded example has less space between twists than the crease pattern shows.*

There are infinitely many ways to apply 442 symmetry to the 4^4 tiling. The only rule is that lines drawn from distinct 4-fold positions to the closest 2-fold position meet at a right angle. This rule is illustrated by the gray guides in the crease patterns of Figure 7(d), with a right angle at the position of 2-fold rotational symmetry of pleats in the hybrid square twist. Leaving this right angle are line segments of equal length that end at the two distinct positions of 4-fold rotational symmetry. These guides may also be used to locate the center of the gray S4 position twist indicated in Figure 7(a) on Figure 3 once the blue S4 and H2 position twists have been chosen, and thus assign the indicated square twist to the gray S4 position in the tiling.

In Figure 7(a), the first 442 symmetry extension on the 4^4 tiling reveals three distinct twist positions, two with 4-fold rotational symmetry and one with 2-fold rotational symmetry that must contain a hybrid square twist. Using a hybrid square twist in this position is the only way to get distinct twists in the two 4-fold positions and these three positions are not independent choices. In fact, they are related by the equation in Figure 7(c)—that the difference in 4-fold position twists along the number line of square twists is determined by the gap of the hybrid square twist in the 2-fold position. Moreover, orientations of hybrid square twists centered below the line of square twist centers in Figure 3 correspond to a shift to the right on the number line of square twists relative to the square twist to the left of the hybrid square. This explains why hybrid orientations 1 and 3 both produce a shift to the right and hybrid orientations 2 and 4 both produce a shift to the left in Figure 7(d). This also tells us that any hybrid twist with the same gap and orientation would equally fill the H2 position, so the choice of blue S4 and H2 positions exactly determines the gray S4 position but the choice of both S4 positions does not determine the exact contents of the H2 position. In this symmetry extension, there are four possible tessellations for each choice of two regular square twists that we saw in the densest symmetry application on this tiling before we even consider the exact choice of hybrid square or spacing.

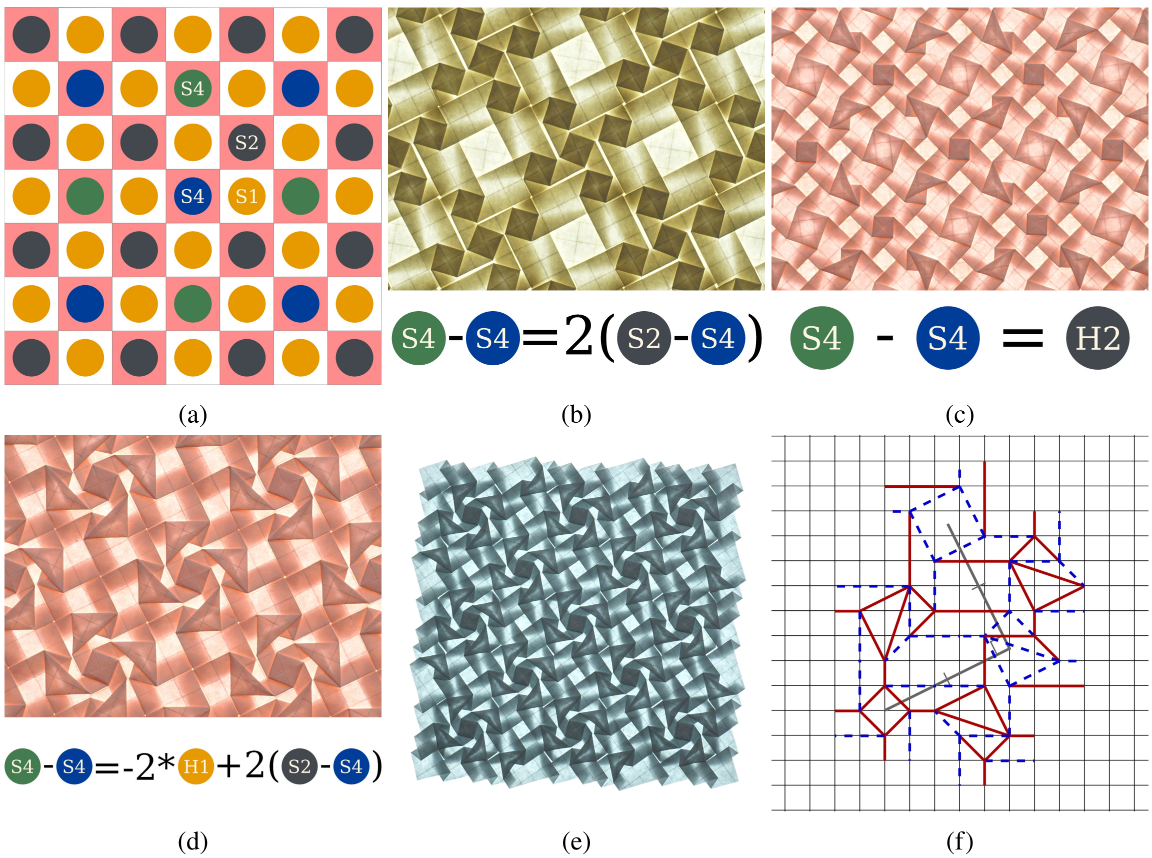


Figure 8: *Second symmetry extension: (a) distinct twist positions and symmetry locations on a* 4^4 *tiling, (b) all regular squares option with Star Crossed design, (c) 2-fold hybrid option with Crossroads design, (d) non-rotational hybrid option with Hybrid Rosettes design, (e) hybrid S1 and S2 option with Lily Pads design, (f) finding the center of the green S4 position of Lily Pads manually, tessellations folded by author.*

The second extension of 442 symmetry on the 4^4 tiling has four distinct positions in the tiling (two with 4-fold rotational symmetry, one with 2-fold rotational symmetry, and one with no rotational symmetry) and four distinct ways to fill these four positions depending on whether hybrid square twists are assigned to the S2 or S1 positions as shown in Figure 8. Either no position, only the S2 position, only the S1 position, or both positions will be filled by hybrid square twists. For the first two ways of filling these positions, the S1 position has no restrictions other than not being a hybrid twist. If you look at this symmetry extension through a combinatorial lens with options only including the closed and open regular square twists and the smallest hybrid square twist (the Closed-Open hybrid), then you have two choices for the first S4 position, eight choices (regular open/closed and front/back or hybrid front/back, horizontal/vertical) for the S1 position, eight choices again for the S2 position, and then the S4 position is fully determined by your previous choices. This gives 128 tessellations, 8 of which belong in the maximum symmetry density because the S1 position was filled by a regular square twist (1/2 of options) and the S2 position choice matched the initial S4 position choice (1/8 of options), and four of which belong in the first symmetry extension. We could reduce the options further by disallowing the intermediating square twist in the second S4 position, but the number of tessellations removed by that restriction is currently unknown. Either way, we have many more options for tessellations to fold with commonly-used twists with each successive symmetry extension.

The design equations for each way of filling the four twist positions available in this second symmetry extension allow me to quickly sort through these many options for ones that will not use the intermediating twist and otherwise fit my subjective artistic criteria. I have not yet deciphered an equation for the case where both the S1 and S2 positions are filled by a hybrid square twist and so must use the manual method for determining the second S4 position as seen in Figure 8(f). The equations follow the same rules as they did in the first symmetry extension in this tiling, with orientations of hybrid twists tied to the labeled positions in Figure 8(a). As mentioned previously, there are infinitely many possible symmetry extensions on the 4^4 tiling, each one with more potential origami tessellations than the last.

Conclusions

The symmetry extensions we explored on the 4^4 tiling give a small taste of what is possible with other tilings as well. The 3^6 and $(3.6)^2$ tilings are particularly amenable to symmetry extensions and I've found usable symmetry extensions and design equations on a tiling that uses hexagon tiles in combination with equilateral triangle and rhombus tiles to form 632 symmetries too. Applying lower symmetries to high-symmetry tilings such as 2222 symmetry to a 4^4 tiling or 333 symmetry to a $(3.6)^2$ tiling and then exploring those symmetry extensions is another fruitful area of exploration for future work.

In summary, origami tessellations can repeat infinitely, there are infinitely many specific twists for any given tile shape, there are infinitely many ways to assign symmetry to a given tiling, and so there are infinitely many twist-based origami tessellations to explore by folding simple twists one at a time.

Acknowledgements

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