Illustrating Transport Theory through Geometric Tiles

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Abstract

In this paper, we show how tools of complex analysis and transport phenomena can be combined with hand-crafted tiling of Euclidean space to create intricate geometric patterns. We showcase a small selection of generated patterns and also show how they can be animated to create interesting visuals.

Introduction

Transformations of space have been a topic of interest to mathematicians and artists alike for quite some time. The work of M.C. Escher, in particular, is permeated by the idea of mapping space onto itself, be it in his intricate perspective sketching, or his famous tilings of the hyperbolic plane, popularized in the mathematical world by Harold Coxeter's book on geometry, among others [4].

More recently, the accessibility of computers have made the explorations of these transformations of space more accessible than ever. Functions can be mapped visually in real-time using freely accessible libraries such as NumPy, and extensive documentation on the mathematics behind these are now available to a wide public on platforms such as Youtube. In this article, I am interested in one specific family of maps of the complex plane onto itself called conformal maps. I first explain briefly what these maps consist of and how they can be used in a wide range of applications of physics. I then show how these maps, as well as other principles of transport theory, can be combined with regular tilings of the plane in order to generate intricate geometric patterns. The patterns thus obtained can then be printed and used as decorative objects, or serve as the basis for further exploration in visual arts. The combination of Escher's work and of conformal transforms has been used in the past as a starting point for people working at the interface between arts and mathematics, for example by reverse engineering the maps used to produce some of his paintings [11], generating intricate tilings of the hyperbolic plane [12], or mapping these same tilings to the interior of arbitrary shapes [8]. The work we present here fits in this tradition, but also has ties to more practical domains such as fluid mechanics and diffusion phenomena, which is my main domain of research. In my research work, I use conformal transforms as tools to solve complex mixing problems. The images presented here were generated by using the same tools I developed in this context, but applied to the generation of aesthetic pictures. Near the end, I also briefly show how I use solutions of transport problems to generate non-conformal maps that can also be of visual interest.

Conformal Maps

Among all possible mappings of the 2D plane onto itself, conformal maps consist of the ones that preserve angles locally. For a pair of function $u(x, y)$ and $v(x, y)$ that map the xy-plane to a new uv-plane, this condition is embodied in the Cauchy-Riemann condition [1]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{1}$$

Functions that respect these conditions are the family of analytic complex functions $f(z)$, where $z = x + yi$, $u(x, y)$ is the real part of $f(z)$ and $v(x, y)$ is its imaginary part. This includes any combination of well-known functions such as $f(z) = z^2$, $f(z) = 1/z$, trigonometric functions, exponentials, etc. Beyond their applications in pure mathematics, complex analysis, and geometry, conformal maps are of special use to physicists, as they can be used to simplify problems related to the Laplace equation $\nabla^2 u = 0$. As such, these transformations have been used extensively in different areas of physics, namely optics [10], heat transfer [3], fluid flow [9], etc.

Applications to Microfluidics

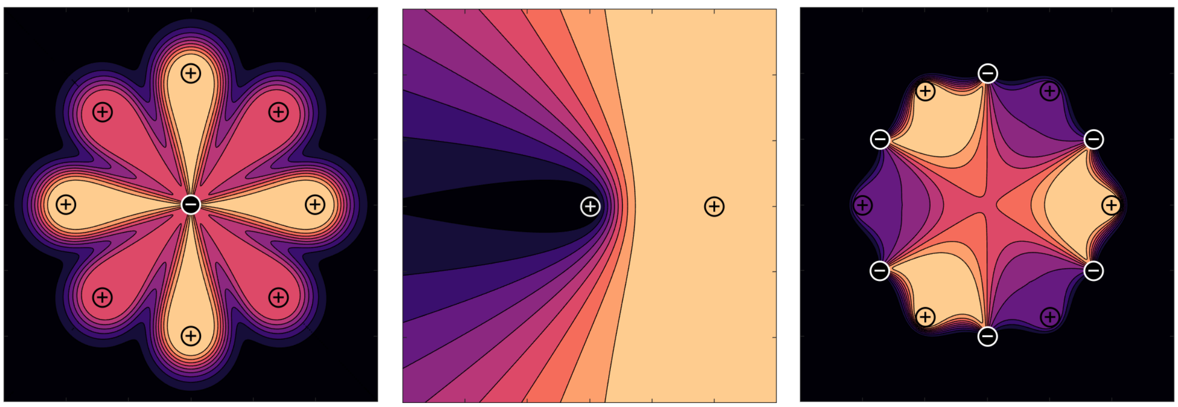


Figure 1: *Solutions to the convection-diffusion equation in the plane. The (+) represent points injecting liquid and the (-) represent points aspirating fluid.*

As part of my research, I use conformal transforms to study diffusion problems in microfluidic systems. Specifically, I study the mathematics of mixing in 2D space where point sources inject and aspirate fluid at different rates. The study of such problems involves solving the convection-diffusion equation using conformal maps. Problems of mixing are first solved in a domain where the differential equation is very simple to solve (for instance in a domain where there is mixing between two miscible substances in a uniform plane flow), and then the results are transformed using conformal maps to complex geometries involving intricate flow. Some examples of concentration profiles obtained in this manner are shown in Figure 1. These results were published in engineering and fluid mechanics journals [6] [2].

An important part of visualizing these solutions was building code that could easily be transform a source solution using simple conformal maps. Once the code was built, it could then be used not only to transform solutions to transport equations, but also any other source image. I started building images made by mapping simple regular tilings using these same transforms, both as a way to visually show the effect of the transforms when presenting my results, and also as a way to generate interesting images.

Mapping Regular Checkerboard Tiles

While the tools of complex analysis required to obtain the solutions in Figure 1 are well-known to mathematicians and physicists, they are often unfamiliar to the engineers I was targeting with my research. As a didactic tool, I thus started using the maps to transform familiar objects: chessboards, maps of the earth, playing cards, etc. The generated images could then be used when presenting to an audience of engineers (or even lay people) to illustrate how specific analytic functions transform the plane.

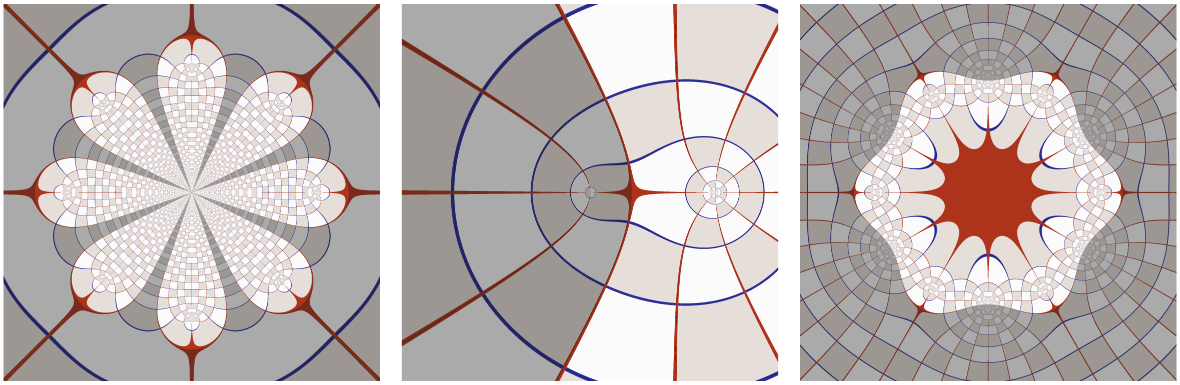


Figure 2: *Maps of simple square checkerboard pattern to illustrate the conformal maps shown in Figure 1. For added readability, the part of the image that corresponds to more concentrated liquid has been made lighter than the rest.*

The solutions presented in Figure 1 were obtained by transforming a simple solution to a partial differential equation (convection-diffusion equation for a regular plane flow) using maps made of linear combinations of logarithms, as well as simple power transforms. The link between the initial simple concentration plot (which is usually just a linear gradient in the vertical direction) and the intricate solutions we end up with is not evident, so in my publications and conference presentations I often presented mappings of simple checkerboard tiles side by side with the "physical" solutions. The mapping of checkerboard tiles has the advantage of immediately highlighting features that have a physical significance, as the horizontal rows map to streamlines (lines that trace the path that the fluid takes), while the vertical columns map to lines of equal potential. Examples of these images are shown in Figure 2. In Figure 2, the streamlines (which are the image of horizontal lines in the initial checkerboard) are colored in red while the equipotentials (which are the image of vertical lines in the initial image) are colored in blue.

Generating More Tiles

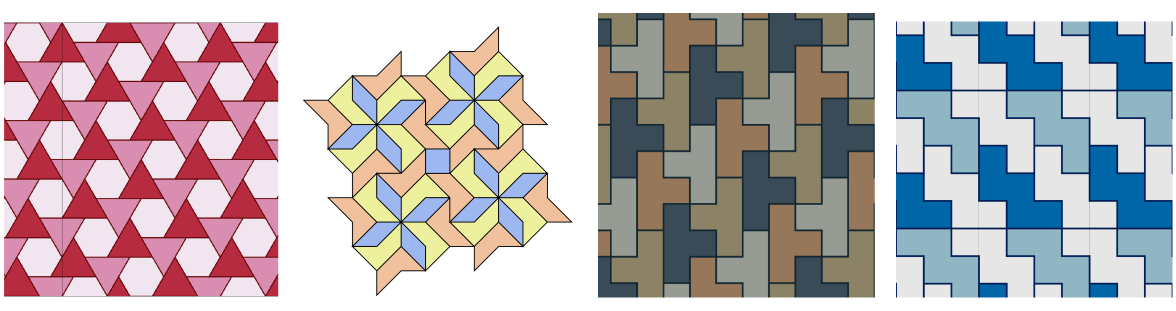
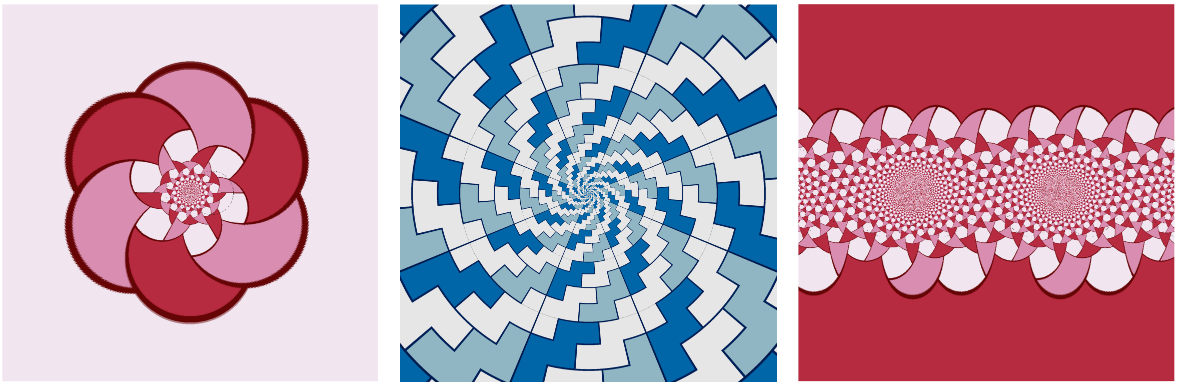


Figure 3: *Tilings that were used to generate the rest of the images in this paper. Colors are kept the same throughout so as to be able to identify which tile was used for which figure*

The next step was to exploit the methodology I had developed to illustrate maps in scientific publications and use it to create different, more interesting images. To do so, I started creating different tilings of the Cartesian planes that I could map in the same way. These tilings were built in inkscape by assembling simple primitive shapes such as squares, equilateral triangles, hexagons, etc. These tilings were all generated by



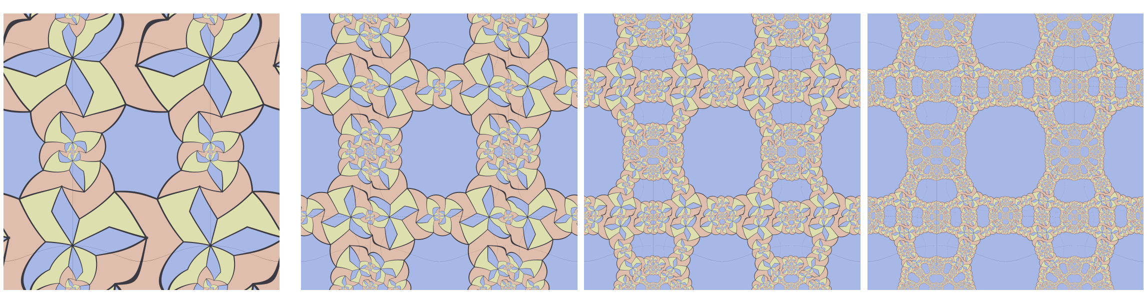
(a) **(b)** **(c)**

Figure 4: *Examples of symmetrical images generated by applying simple conformal maps to regular tilings.*

hand, without needing an automated or systematic way to do so. The objective was always to strike a balance between visually interesting patterns, and patterns that were simple enough to properly illustrate the effect of the map. The choice of color is done either by picking complementary colors, or by sampling from an interesting photograph, and the tiles are colored so as to highlight interesting features of the tiling. Tiling of the Cartesian plane is a very vast subject [7], and much future work could be done at this step to generate more intricate images. Some source tiles that were used to generate the images in the rest of this paper are shown in Figure 3.

Some examples of these tiles transformed by using maps described by simple analytic functions are shown in Figure 4. Figure 4(a) shows the hexagonal tiling mapped using an inversion transform $f(z) = \frac{1}{z}$, Figure 4(b) shows the staircase tiling mapped using a logarithm function $f(z) = \log(z)$, and Figure 4(c) shows a hexagonal tiling mapped using a tangent function $f(z) = \tan(z)$. Even when using very simple functions, very intricate geometric patterns can start to appear. For example, the long diagonals that appear in the staircase tile map to spirals when a logarithm transform is applied (see Figure 4(b)). Some of the most interesting patterns happen around the singularities of the maps, such singularities appear in Figures 4(a) and 4(c).

Schwarz-Christoffel Transforms

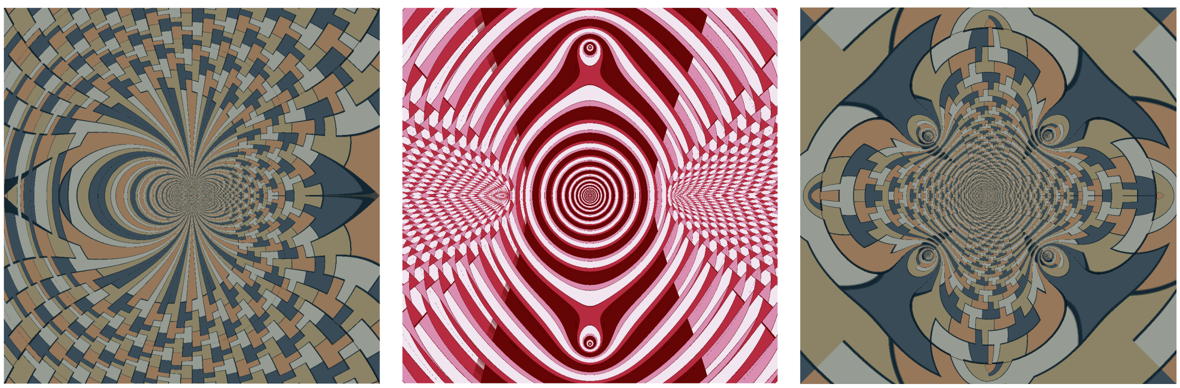


(a) **(b)** **(c)** **(d)**

Figure 5: *Mapping of tiles done using elliptic functions. This maps the entirety of the complex plane within a square tile which can then itself be used to tile the plane. Subfigures (b), (c), and (d) were obtained by composition of the same elliptic function with itself respectively 2, 3, and 4 times.*

An interesting family of conformal maps is Schwarz-Christoffel transforms [5], which are maps of the upper half of the complex plane to the interior of polygons. In my everyday microfluidics work, I use Schwarz-Christoffel transforms to obtain solutions to problems of mixing inside rectangular chambers, long channels, channels with corners in them and other domains that are polygonal in shape. The mapping of the upper half-plane to the interior of squares and rectangles is given in terms of elliptic integrals (specifically, the map to the interior of a square is the Jacobi Elliptic Sine). These maps from the half-plane to the interior of squares and rectangles are the most complex Schwarz-Christoffel maps for which you can get a closed form, for polygons with more sides, numerical methods are needed. Here we fill the entirety of the complex plane with an initial tiling, then map it to the interior of a square using a Jacobi Elliptic Sine. This map has the advantage that the resulting image is itself a square that can tile the plane. This new tile can be taken as a starting point for generating a new image. This iterative procedure of applying the elliptic function successively is illustrated in Figure 5, where we show the result of repeatedly applying a map to an initial tile.

Non-Conformal Maps: Concentration Plots



(a) **(b)** **(c)**

Figure 6: *Mapping of tiles done by combining regular conformal maps and solutions of the convection-diffusion equation*

Beyond just using analytic functions to plot geometric patterns, we can exploit the solution patterns obtained in Figure 1 to generate non-conformal maps. In the geometries shown there, the concentration field (represented by the color hue) varies in a direction that is approximately perpendicular to the streamlines (or horizontals in the original map). This means that lines of equal potential and lines of equal concentration should be approximately at right angles from each other, generating a grid that could be used as a mapping. In such a map, rather than mapping the horizontal coordinate x to the real part of a complex function, and the vertical coordinate y to its imaginary part, like we did in the rest of the paper, we map the horizontal coordinate x to the real part of the function, and the vertical coordinate y to the level sets of the associated diffusion problem (examples of which are shown in Figure 1). This creates a grid not be made of squares but rather of deformed rectangles, which can get very thin. Such a map runs the risk of severely distorting the tile pattern in some areas, which could make the images unintelligible. However, by playing around with aspect ratios, it is possible to use such a map to obtain cool outputs, a sample of which is shown in Figure 6. In some of these figures, the interplay between the severe distortion in some regions of space, and the absence of it in others, creates an interesting contrast. For each of the images in Figure 6, the horizontal coordinate maps to the imaginary part of a function $f(z)$, while the vertical coordinate maps to a function of the form $\mathrm{erf}\left(\sqrt{f(z)}\right)$. The functions $f(z)$ are sums of logarithm functions; for example, Figure 6(a) was produced by the function $f(z) = \log(z) - 2\log(z-1)$. These functions have applications in the modeling of flow from a set of point sources, such as those illustrated in Figure 1, but here they are just used to generate distorted pictures.

Animating Results

In addition to static images, I have also generated animations by either continuously translating the original tile, or slowly rotating it until it has done a full revolution. Such transformations of the original tiling eventually return to their original position, which allows me to create continuously looping animations. Examples of such animations are provided as supplementary material.

Summary and Conclusions

In conclusion, I have shown how various transformations of space can be used to generate intricate geometric patterns, and how these can then be animated. The generated pictures have an aesthetic value in and of themselves but they could also be used as sources for further work in more traditional media, for instance as templates for ink drawings or for wood block printing, which are steps I hope to accomplish in the near future.

Acknowledgements

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