Brownian Bridges on Polygons

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Abstract

There are no straight lines or sharp corners in nature, said the famous Catalan architect Antoni Gaudí. The famous Polish–French–American mathematician Benoit B. Mandelbrot went even further and asserted that the curves in nature are fractal. Inspired by Mandelbrot's assertion, we consider regular shapes with edges replaced by stochastic fractals. In particular, we consider equilateral triangles, squares, and pentagons where the edges are replaced by realizations of different Brownian bridges: a "normal" Brownian bridge, a reflective Brownian bridge, and a sticky Brownian bridge.

Brownian Motions and Bridges

The Brownian motion [2], or the Wiener process, $W = (W_t)_{t\in[0,1]}$ is arguably the most important stochastic process there is. One physical interpretation of the Brownian motion is the motion of a "Brownian particle", e.g. pollen, under the bombardment of much smaller particles, e.g. water molecules. Indeed, the Brownian motion is named after Scottish botanist Robert Brown who studied in [3] the movement of pollen in water.

The Brownian motion can also be defined to be the continuous time limit of random walks. A more rigorous mathematical definition of the Brownian motion is that it is the unique process that has stationary and independent increments so that the increments $W_t - W_s$ have normal distribution with mean 0 and variance $\sigma^2|t-s|$ for some fixed $\sigma^2 > 0$.

For our purposes it is enough to note that the Brownian motion on the baseline $[0, 1]$ sampled on the equidistant points $t_k = k/N$, $k \in \{1, \ldots, N\}$, can be realized as $W_0 = 0$ and

$$W_{t_k} = \frac{\sigma}{\sqrt{N}} \sum_{j=1}^{k} \xi_j,$$

where the ξ_j's are independent standard normal random variables. Here $\sigma > 0$ is the scale of the Brownian motion: $\sigma^2 = \mathbb{V}\mathrm{ar}(W_1)$.

The Brownian motion is $1/2$-self-similar meaning that for each $a > 0$ the scaled process $a^{-1/2}W_{a\cdot}$ and the original Brownian motion W have the same distribution. Consequently, the Brownian motion is fractal in the sense that its paths have Hausdorff dimension $3/2$.

The Brownian bridge $B = (B_t)_{t\in[0,1]}$ is the Brownian motion conditioned so that $\{B_1 = 0\}$. The Brownian bridge can be constructed from the Brownian motion as

$$B_t = W_t - tW_1.$$

This means that the Brownian particle is conditioned to return to the baseline at time 1 forming a "bridge".

The reflective Brownian bridge $R = (R_t)_{t\in[0,1]}$ is defined by reflecting the Brownian bridge with respect to its baseline:

$$R_t = |B_t|.$$

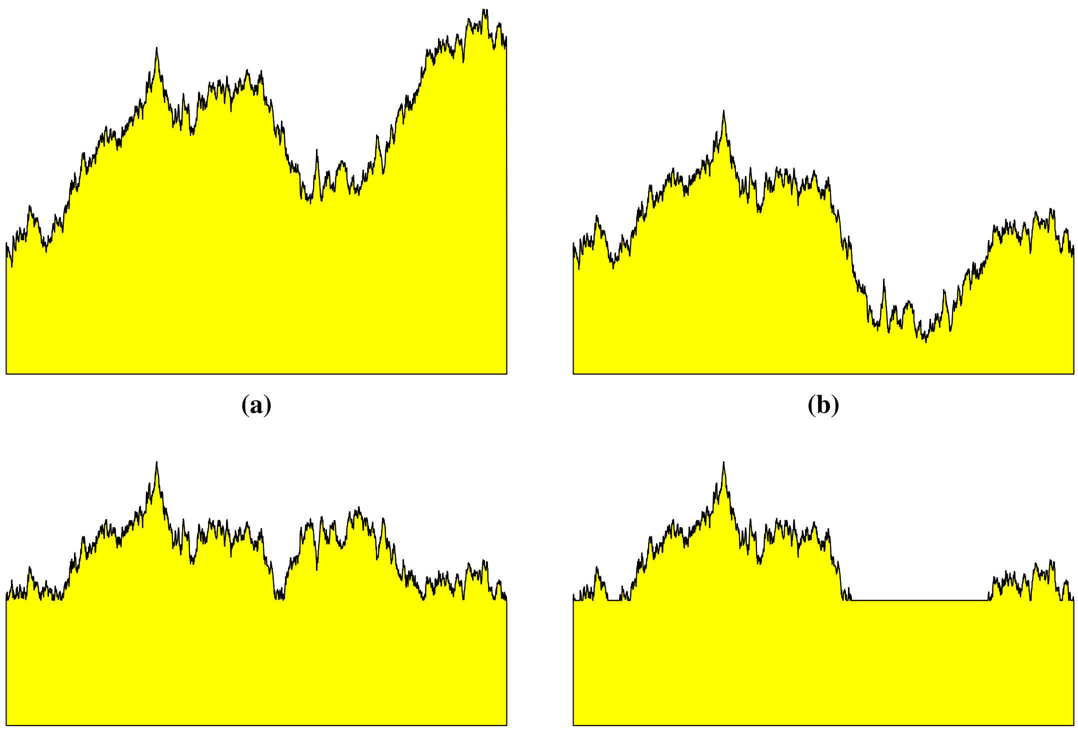
This means that the Brownian bridge particle is reflected in the baseline instead of allowing it to cross it.

The sticky Brownian bridge $S = (S_t)_{t\in[0,1]}$ is defined by gluing the negative part of the Brownian bridge onto the baseline:

$$S_t = \max(B_t, 0).$$

This means that the Brownian particle remains on the baseline when the "free" Brownian bridge particle would be below the baseline.

In Figure 1 we illustrate the Brownian motion, the Brownian bridge, the reflective Brownian bridge, and the sticky Brownian bridge with "torn paper" images. The processes were sampled on $N = 2048$ equidistant time points by using GNU Octave [4]. The pictures were produced by Asymptote vector graphics language [1]. All the codes are available from the author upon request.

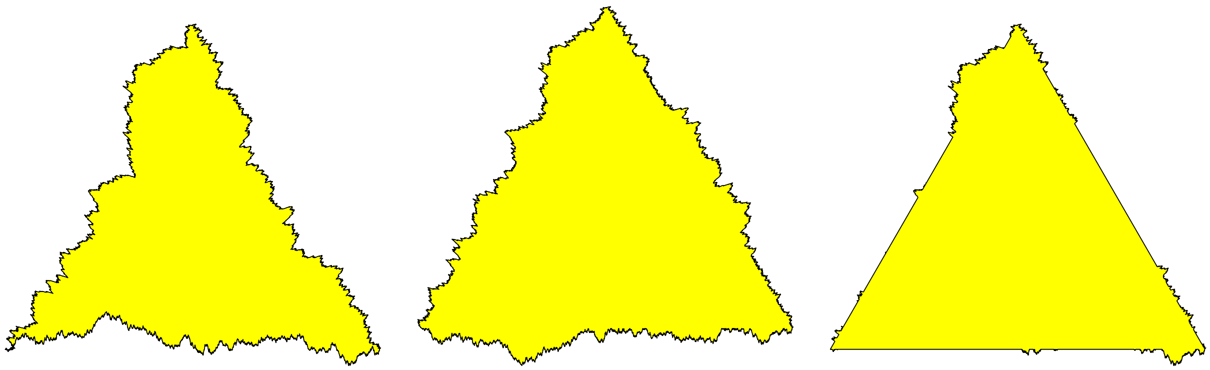


(a) **(b)** **(c)** **(d)**

Figure 1: *"Torn papers": with scale* $\sigma = 0.1$ *(a) Brownian motion, (b) Brownian bridge, (c) Reflective Brownian bridge, (d) Sticky Brownian bridge.*

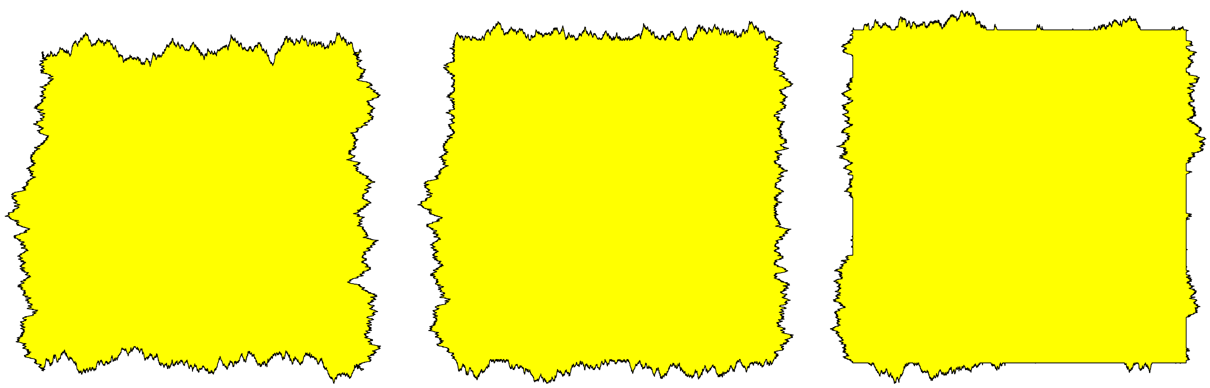
Triangles, Squares, and Pentagons

In the following pictures the edges of equilateral triangles, squares, and pentagons were replaced by Brownian bridges, reflective Brownian bridges, and sticky Brownian bridges by taking the edges to be the unit-length baseline. We could have used any polygons, the choice of equilateral triangles, squares, and pentagons just felt natural. The pictures were produced by Asymptote vector graphics language. The processes were sampled on $N = 2048$ points for each edge of the polygon. The samples were produced with GNU Octave. All the codes are available from the author upon request.



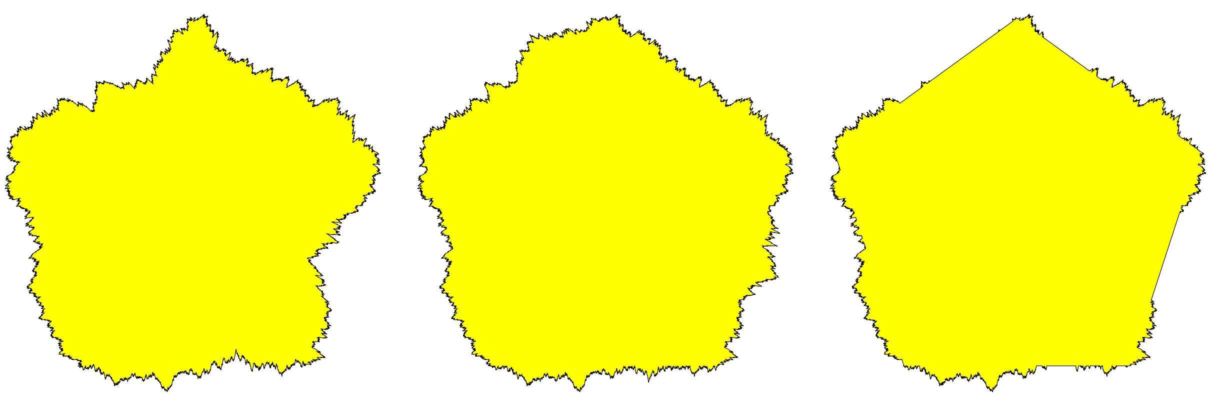
(a) **(b)** **(c)**

Figure 2: *Brownian bridges with scale* $\sigma = 0.01$ *on equilateral triangles: (a) normal Brownian bridge, (b) Reflective Brownian bridge, (c) Sticky Brownian bridge.*



(a) **(b)** **(c)**

Figure 3: *Brownian bridges with scale* $\sigma = 0.015$ *on squares: (a) normal Brownian bridge, (b) Reflective Brownian bridge, (c) Sticky Brownian bridge.*



(a) **(b)** **(c)**

Figure 4: *Brownian bridges with scale* $\sigma = 0.02$ *on pentagons: (a) normal Brownian bridge, (b) Reflective Brownian bridge, (c) Sticky Brownian bridge.*

Further Work

The idea presented here can be further developed into many directions: (a) one can consider more general shapes than polygons, even fractal shapes like the Koch snowflake, (b) one can consider drawing the boundary with different iterations (N) and/or scales (σ) of the process with different colors, and (c) one can consider more general Gaussian processes than the Brownian motion, e.g. the fractional Brownian motion. Finally, (d) one and consider tessellations with Brownian bridge boundaries.

Acknowledgements

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References

[1] *Asymptote: the Vector Graphics Language*. https://asymptote.sourceforge.io/.

[2] *Brownian motion*. https://en.wikipedia.org/wiki/Brownian_motion.

[3] R. Brown. “A brief account of microscopical observations made in the months of June, July and August, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies.” *Philosophical Magazine*, 1828.

[4] J. W. Eaton, D. Bateman, and S. Hauberg. *GNU Octave : a high-level interactive language for numerical computations*. Network Theory, 2008.