Trisection of an Angle

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Abstract

Dividing an angle into thirds is one of the three classic problems that cannot be solved with ruler and compass, along with doubling the cube and squaring the circle. We discuss a very simple fit-in solution using squares.

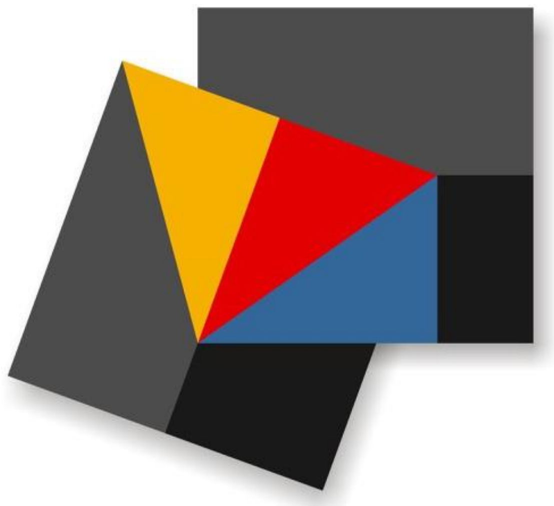
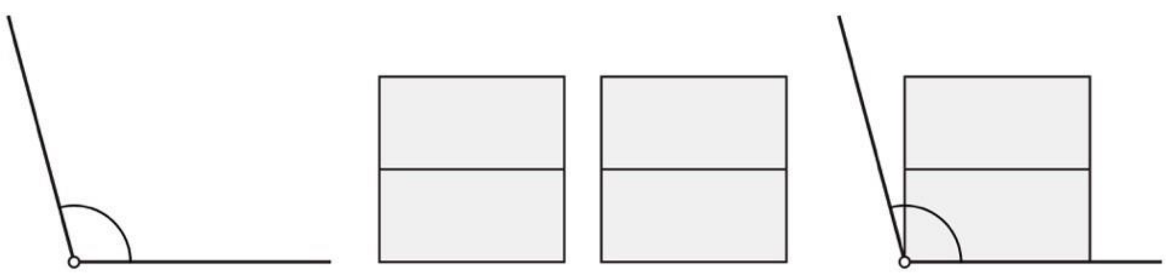


Figure 1: *Jo Niemeyer artwork 2022.*

Introduction and Procedure

This paper describes a new approach to the classical angle trisection problem. The approach is related to the use of the "tomahawk" tool as described in [3] and the paper folding technique used in [1]. The novelty of this related approach is to use two transparent squares as tools in the construction. Figure 1 presents an artwork by the first author based upon the constructive proof described below. See also [2] and [4] for other geometrical artworks by the authors.

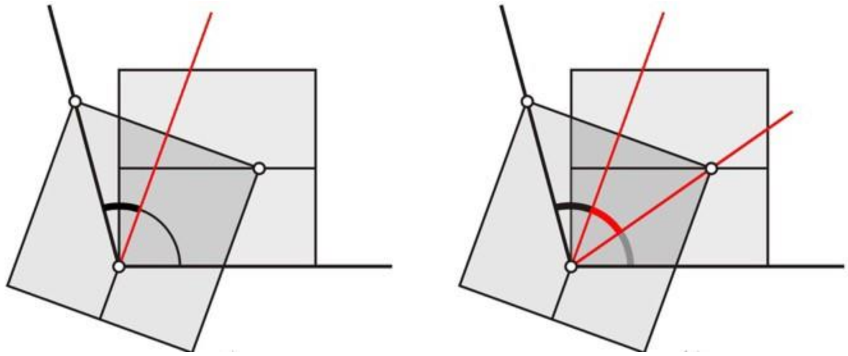
We work with two squares made of transparent material, each with a central parallel midline (Figure 2b). We get the midline simply by folding. We place one square at one of the legs of the given angle as shown in Figure 2c. The midline of the square is parallel to the leg.



(a) (b) (c)

Figure 2: *Laying out the square.*

We now insert the second square in such a way that one vertex of the square lies on the other leg of the angle, the vertex of the square that is mirrored in relation to the midline lies on the midline of the first square and the midline itself runs through the vertex of the angle (Figure 3a). The midline of the second square is one of the two lines trisecting the given angle (Figure 3a).

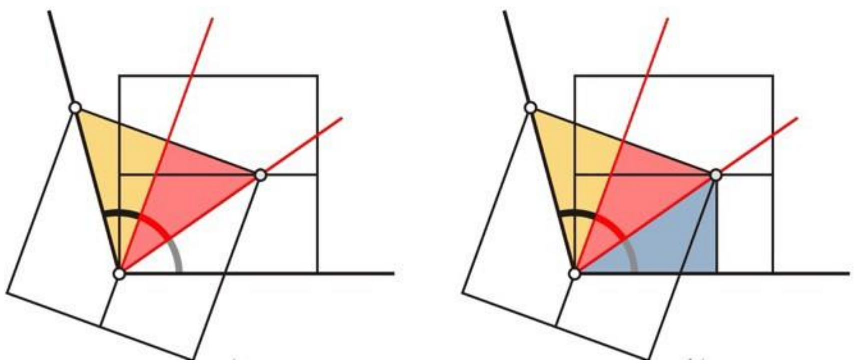


(a) (b)

Figure 3: *Trisection of the angle.*

The second line trisecting the angle is the line from the vertex of the angle to the insertion point on the midline of the first square (Figure 3b). If the given angle is less than 3*arctan(1/2) ≈ 79.695°, or greater than 270º, we have to proceed with elongated squares, i.e., with rectangles.

Proof of the Trisection



(a) (b)

Figure 4: *Proof.*

To prove the trisection, we use the three marked right triangles (Figure 4). These three triangles are congruent, as follows. The yellow and red triangles are mirror images (reflection about the midline of the second square) and therefore congruent. The red and the blue triangle each have a right angle, equal short legs of the right angle (half the side length of the squares) and the hypotenuse in common. They are therefore congruent. Hence all three triangles are congruent. At the vertex of the given angle, we now have three equal angles.

References

[1] Thomas Hull. *Project Origami: Activities for Exploring Mathematics, Second Edition*. AK Peters/CRC Press, 2012.

[2] Jo Niemeyer. Website. http://www.joniemeyer.com/2022/index.html.

[3] Tomahawk (geometry) https://en.wikipedia.org/wiki/Tomahawk_(geometry).

[4] Hans Walser. Website. https://www.walser-h-m.ch/hans/.