Transition Processes for Frieze Patterns

D. Jacob Wildstrom

Department of Mathematics, University of Louisville; djwild01@louisville.edu

Abstract

The seven frieze patterns are well-known but if one wants to exhibit multiple symmetries in a single strip of material, transitions between them are necessary in order to showcase all available symmetries without abrupt changes of design. This work explores the viable transformations among the seven conventional frieze patterns, with preliminary consideration of extending this result to the seventeen frieze patterns which make use of color-reversal symmetries.

It is well known that there are seven different symmetry groups possible for a periodic pattern appearing in a strip (frequently called "frieze patterns"). Many papers have discussed frieze patterns particularly in consideration of their cultural prevalence in non-representational art [1, 4, 5, 6], and they are admirably suited for decorative borders or narrow strips of material such as belts. To exhibit multiple patterns in a single crafted work, however, the use of transitional states which allow one pattern to naturally flow into another provides an unobtrusive means to use a variety of different symmetry groups within a single work. Transitions between different tilings or tessellations have been used extensively in tiling-oriented work, particularly in M.C. Escher's *Metamorphosis* series of prints and in William Huff's parquet deformations, which have also been the subject of previous Bridges papers[2, 3]. Most of the work involving pattern transitions has, however, involved deformations of a pattern while preserving a single underlying symmetry, rather than demonstrating a variety of symmetries; a notable exception is Karl Schaffer's explorations of symmetry in dance, where change over time of the symmetries among groups of dancers is a widely used choreographic technique[7]. This paper seeks to illuminate how multiple different symmetries could be expressed within a single work with suitable transitions: for instance, while the frieze groups could be illustrated with seven individual belts, each with a different pattern, an elegant means to exhibit symmetries would allow exhibition of all symmetries with a single belt.

Because this project is intended as a preface to an eventual exploration specifically of the color-reversal symmetries discussed above, we can establish at the outset that it is desirable for the patterns to have an equal amount of positive (black) and negative (white) space; this is a necessary property of any color-reversal symmetries, but we will require it even though this work concerns patterns where it is not mandated by the underlying symmetries. Also in consideration of later bringing this larger class of symmetries into play, in order to properly consider the seven standard frieze symmetry groups as constituting subgroups of the color-reversal groups, we will exhibit as representatives of these seven frieze patterns only those which do not possess any color-reversal symmetries at all.

Subject to these restrictions, representative color-balanced patterns for the seven frieze groups are shown in Figure 1 and identified by their standard crystallographic names. One symmetry might be said to be of

$p111$ $pm11$ $p1m1$ $p1g1$ $p112$ $pmg2$ $pmm2$

Figure 1: *Seven patterns exhibiting the possible frieze symmetries*

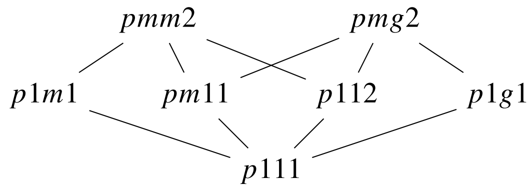


Figure 2: *Hasse diagram for the seven frieze symmetries*

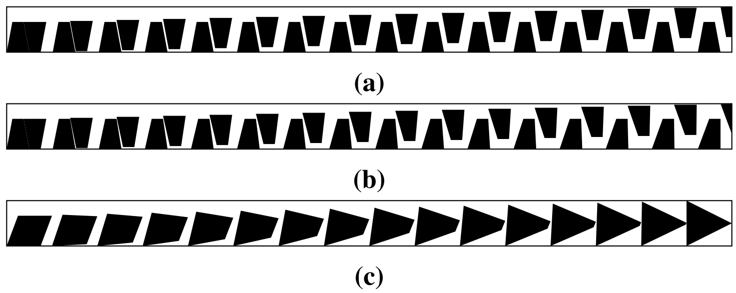


Figure 3: *Transformations of patterns from* $p111$ *symmetry to (a)* $pmg2$ *symmetry, (b)* $p1g1$ *symmetry, and (c)* $p1m1$ *symmetry*

lower-order than another if its group of pattern-preserving transformations is a subgroup of the higher-order symmetry. The Hasse diagram of the poset of symmetry-preserving transformations can be seen in Figure 2.

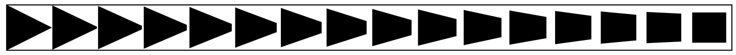
By a *transition* from a lower to a higher-order symmetry, we mean a continuous deformation of the pattern appearing in a single period such that the pattern exhibits the lower-order symmetry at all points during the transformation, the higher-order symmetry only at the end of the deformation, and no other symmetries at any point during the deformation. There are twelve comparable pairs of symmetries in Figure 2, noting specifically that the comparability between $pmm2$ and $p111$, and between $pmg2$ and $p111$, are omitted from the Hasse diagram as they are implied by transitivity of order. It would be ideal, from a point of view of exhibiting all these comparabilities, if it was possible to produce transitions for each pair of comparable symmetries.

Transitions Among Frieze Symmetries

The specific patterns shown in Figure 1 are such that several of the transitions are straightforward invocations of well-known geometric transformations, which for the purpose of simplicity we will describe as being applied to the black regions. Specifically, the pattern exhibited for $p111$ can be transformed into the pattern exhibited for $pm11$ by shearing the parallelograms into rectangles, and a similar process will transform the $p112$ pattern into $pmm2$. In addition, $p111$ can be transformed into $p112$ by translating the parallelograms vertically to the center-line and a similar process performed on the rectangles transforms $pm11$ into $pmm2$. Finally, $p111$ can be transformed directly into $pmm2$ by performing both of these transformations simultaneously. A similarly straightforward transition on the examples given is the result of shearing the trapezoids shown for $p1g1$ into the isosceles trapezoids of $pmg2$. For purposes of brevity, none of these simple transitions are exhibited in full.

Because $p111$ is a very permissive symmetry, free deformation of the $p111$ pattern into the three patterns for which its transition has not yet been described ($p1m1$, $p1g1$, and $pmg2$) are also quite straightforward, although deformation in an area-preserving manner into $p1m1$ is arithmetically somewhat messy. In particular, the $p111$ to $pmg2$ transition can be easily obtained by bisecting each parallelogram into two trapezoids,

(a)



(b)

(c)

Figure 4: *Transformations of patterns from (a)* $pm11$ *to* $pmg2$ *symmetry, (b)* $p1m1$ *to* $pmm2$ *symmetry, and (c)* $p112$ *to* $pmg2$ *symmetry*

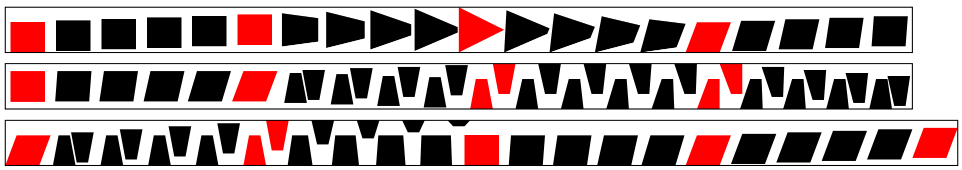


Figure 5: *A frieze of 61 repetitions, undergoing all twelve transitions; transition endpoints are shown in red.*

and translating one of them upwards and rightwards. This transformation is exhibited linearly in Figure 3a. The transformation to $p1g1$ is quite similar, incorporating a shear element as seen in Figure 3b. To transform to $p1m1$, the two rightmost vertices of the parallelogram can be freely moved towards the right vertex of the triangle while the upper left vertex moves further up and left; although two of those motions are linear, in order to preserve area the lower right point will in fact trace out a parabolic path. This transform can be seen in Figure 3c.

The graph in which each symmetry is represented by a vertex and valid transitions are represented by adjacencies is very nearly visually identical to the Hasse diagram in Figure ; the only difference is that the comparability that both $pmm2$ and $pmg2$ have to $p111$, which is left implicit in a Hasse diagram, would be an edge in the graph of transitions. Eulerian or Hamiltonian traversals on this graph would represent showcases respectively of every transition and of every individual symmetry.

Although an Eulerian circuit is not possible on the graph of transitions, as both $pm11$ and $p112$ participate in three transitions each, an Eulerian trail from $pm11$ to $p112$ on the graph of transitions is possible, which means it is possible to make a strip of material which exhibits all the possible transitions between states. Such a strip, with each transition consisting of five repetitions per transition, is illustrated in Figure 5. This particular strip corresponds to the Eulerian trail which in turn visits the vertices $pm11$, $pmm2$, $p1m1$, $p111$, $pmm2$, $p112$, $pmg2$, $p1g1$, $p111$, $pmg2$, $pm11$, $p111$, and $p112$.

Period modification

It is notable that any symmetry with a horizontal axis of reflection does have a glide reflection, but a glide reflection whose associated translation distance is a full period, rather than half of one. There is thus a sense in which the Hasse diagram in Figure 2 is not entirely accurate: $p1m1$ and $p2mm$ have glide-reflection symmetries and thus their symmetry groups are a superset of the symmetry groups of $p1g1$ and $pmg2$, albeit on a period twice as large. A fuller exploration of transitions among symmetries would thus consider relationships not only among symmetry groups of a single period, but also considering their relationships with symmetry groups on periods whose length has been halved or doubled.

If we denote the symmetry groups on a half period with a subscripted 2, then we could consider an expanded poset of the fourteen full-width and half-width symmetries where in addition to the established orderings within the full-width subposet and the half-width subposet, and the comparability between each full-width symmetry and its half-width variant, we would also have the comparabilities $p1g1 < p1m1_2$ and $pmg2 < pmm2_2$. This structure could be extended indefinitely upwards or downwards, with subposets for not only a halved period, but also quartered or smaller, and in the opposite direction, for doubled, quadrupled, and so forth.

Future work

There are many other families of symmetries on which the question of designing transitions between the symmetries may be of interest for building a showcase work exhibiting multiple symmetries. This project was originally born out of a desire to build a design a woven belt showcasing all the seventeen frieze antisymmetries systematized by Weber [8], which would depend on a plan to exhibit all of the symmetries via a Hamiltonian traverse on the transition graph for frieze antisymmetries, or more ambitiously an Eulerian traverse to exhibit all the antisymmetry transitions. This same process of identifying transitions may also be of relevance to the seventeen planar symmetries, also known as the "wallpaper patterns", or the forty-six "counterchange" plane symmetries of H.J. Woods [9]. Transitions in plane symmetries, however, present distinct challenges and opportunities: not only the patterns but also the fundamental domains must change shape as part of the transition, and the ability to transition a pattern in two different directions along the two different axes means that the underlying structure of a work undergoing transitions might be richer than simply a walk through a graph.

References

[1] B. L. Bodner. "Frieze Patterns of the Alhambra." *Bridges Conference Proceedings*. San Sebastian, Spain, July 24–27 2007. pp. 203–208. http://archive.bridgesmathart.org/2007/bridges2007-203.html.

[2] C. S. Kaplan. "Metamorphosis in Escher's Art." *Bridges Conference Proceedings*. Leeuwarden, the Netherlands, July 24–28 2008. pp. 39–46. http://archive.bridgesmathart.org/2008/bridges2008-39.html.

[3] C. S. Kaplan. "Curve Evolution Schemes for Parquet Deformations." *Bridges Conference Proceedings*. Pécs, Hungary, July 24–28 2010. pp. 95–102. http://archive.bridgesmathart.org/2010/bridges2010-95.html.

[4] L. Koss. "One-color Frieze Patterns in Friendship Bracelets: A Cross-Cultural Comparison." *Bridges Conference Proceedings*. 2021. pp. 253–256. http://archive.bridgesmathart.org/2021/bridges2021-253.html.

[5] G. R. Laigo, H. M. GebreYohannes, and F. M. H. Al Khamisi. "Symmetry Groups of Islamic Patterns at the Sultan Qaboos Grand Mosque." *Bridges Conference Proceedings*. 2014. pp. 183–190. http://archive.bridgesmathart.org/2014/bridges2014-183.html.

[6] L. Radovic and S. Jablan. "Antisymmetry and Modularity in Ornamental Art." *Bridges Conference Proceedings*. 2001. pp. 55–66. http://archive.bridgesmathart.org/2001/bridges2001-55.html.

[7] K. Schaffer. "Dichromatic Dances." *Bridges Conference Proceedings*. Waterloo, Canada, July 27–31 2017. pp. 291–298. http://archive.bridgesmathart.org/2017/bridges2017-291.pdf.

[8] L. Weber. "XII. Die Symmetrie homogener ebener Punktsysteme." *Zeitschrift für Kristallographie*, vol. 70, 1929, pp. 309–327. https://doi.org/10.1524/zkri.1929.70.1.309.

[9] H. J. Woods. "The Geometrical Basis of Pattern Design. Part IV: Counterchange Symmetry in Plane Patterns." *Journal of the Textile Institute Transactions*, vol. 27, no. 12, 1936, pp. T305–T320. https://doi.org/10.1080/19447023608661695.