

# Alexandrov Puzzle

Kodai Takenaga<sup>1</sup> and Shizuo Kaji<sup>2</sup>

<sup>1</sup>Dept. of Mathematics, Kyushu University, Fukuoka, Japan; koudai.980723@gmail.com

<sup>2</sup>Inst. of Mathematics for Industry, Kyushu University, Fukuoka, Japan; skaji@imi.kyushu-u.ac.jp

## Abstract

A. D. Alexandrov proved a remarkable theorem that gives a necessary and sufficient condition for a simple polygonal figure on a plane to be folded into a convex polyhedron by glueing its edges. In some cases, a single figure admits multiple ways to glue its boundary to yield different convex polyhedra. A few algorithms have been developed to enumerate glueing patterns on a fixed polygonal figure that satisfy Alexandrov's condition. Based on these facts, we introduce a physical puzzle consisting of hinged panels that can be folded into different polyhedra. The puzzle challenges one's three-dimensional perception with a tangible object.

## Introduction

Everyone knows that the Latin cross can be folded into a cube, while few have ever imagined folding it into a different shape. In fact, the Latin cross is known [5] to be folded into four other convex polyhedra shown in Figure 1; here, we use the word convex polyhedra to include degenerate convex figures such as a flat doubly-covered quadrilateral. This interesting phenomenon is related to a theorem of Alexandrov, who is the namesake of our *Alexandrov puzzle*. Our puzzle consists of polygonal panels made of thick paper that are joined by masking tape along the edges that serve as hinges (Figure 1). In the unfolded state, it has the shape of the Latin cross on the plane with internal edges which can flex by hinges. The goal is to find the five polyhedra by glueing each of the fourteen edges of the Latin cross to another edge.



*Alexandrov puzzle in the unfolded state*

*(Q) Doubly-covered quadrilateral*

*(T) Tetrahedron*

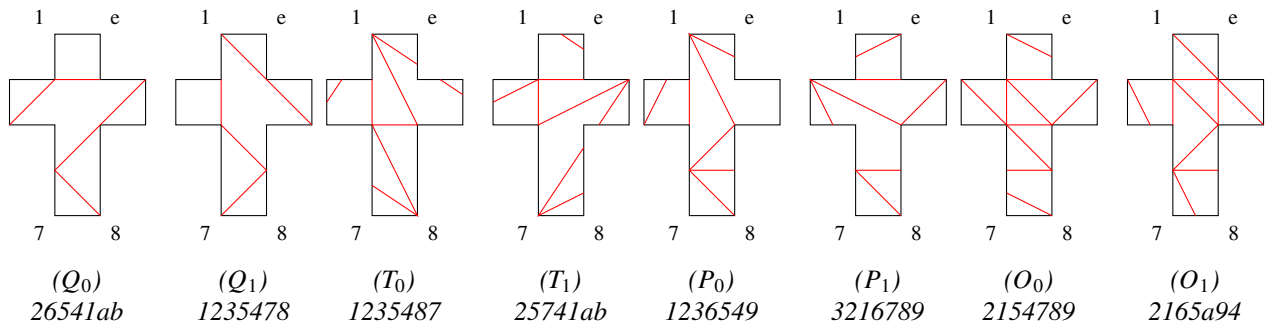
*(P) Pentahedron*

*(O) Octahedron*

**Figure 1:** *The Alexandrov puzzle and the four solutions except for the cube.*

Alexandrov's theorem [1] states that any convex polyhedral metric admits a realisation as a polyhedron unique up to Euclidean motion. A brief history and accessible account of this deep result are found in [3, §23]. According to this theorem, a simple polygonal figure on a plane can be folded into a convex polyhedron if and only if the boundaries are so glued that it has the topology of a sphere and the angles sum up to  $2\pi$  or less at each vertex. The theorem also guarantees that the fold lines (crease pattern) are uniquely determined once a valid glueing pattern is given. It should be noted that glueing lines can lie on faces of the resulting polyhedron.

Despite the uniqueness, it is often difficult to find the fold lines for a given glueing pattern. To solve our puzzle, valid glueing patterns and the corresponding fold lines should be found at the same time. The puzzle has recreational and educational merits for a wide audience, from school kids to professional mathematicians. The instructions on how to produce the puzzle, from readily available materials to the computer program needed to enumerate all possible puzzle configurations, are available at [6].



**Figure 2:** Crease patterns together with the corresponding zipping sequences. The vertices are named from 1 to 9 and a to e in the counterclockwise manner. Each polyhedron has two essentially distinct crease patterns, which are distinguished by the subscript 0 and 1.

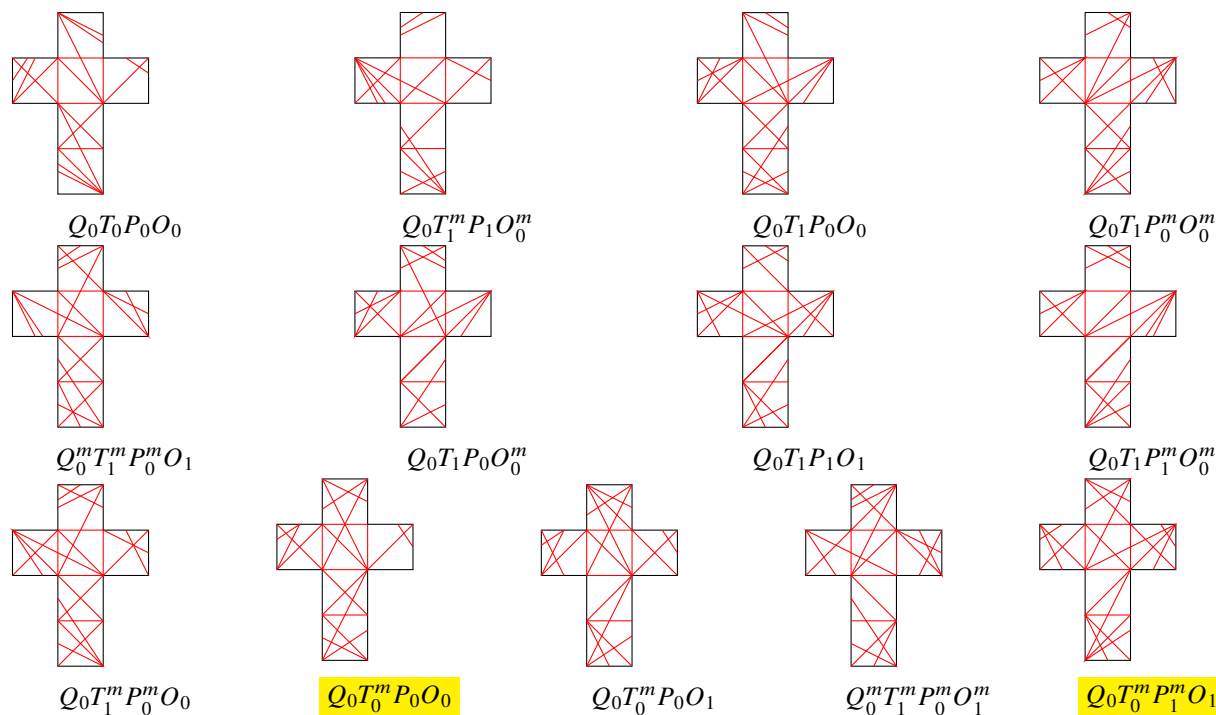
### Convex Polyhedra Foldable from the Latin Cross

We take the Latin cross as a concrete example throughout this paper although a similar puzzle can be made for other polygonal figures. The five folding (the four shown in Figure 1 and the cube) of the Latin cross is mentioned in [5] and made into a video in [2]. An enumeration using a computer program shows that there are exactly 23 distinct polyhedra, including degenerate ones [3, §25.6]. Among the 23 polyhedra, the five are exactly those which can be obtained by *edge-to-edge glueing*, which glues each edge of the boundary of the Latin cross to another edge. The glueing can be specified by a sequence of operations called the *zipping* ([3, §25.1.2]). Given a vertex  $x$  in the boundary of the Latin cross, the *zipping* at a base point  $x$  is obtained by glueing the two points on the two boundary edges adjacent to  $x$  that have the same distance from  $x$ . The resulting partially-glued shape has a boundary with two edges less than the original. We can continue the process until there are no boundary edges left. This process is recorded by the sequence of the base points. The presentation of the glueing by a sequence is not unique; for example, consecutively zipping at two distant points would be commutative. All valid edge-to-edge glueing patterns and their minimum sequences with respect to the lexicographic order are listed in Figure 2, which is obtained by our computer program ([6]).

### Construction of Alexandrov Puzzle

For a comfortable trial-and-error experience, the boundary should be easily closed and opened. Magnets or velcro (little black squares in Figure 1) are attached to the boundary. It is notable that by assigning the S and N poles of magnets (or the hoop and loop sides of velcro) alternately on the boundary edges of the Latin cross, we can glue the boundary by matching S and N poles. More precisely, the following holds: If we number the boundary of a polygonal figure sequentially with the natural numbers, the glued edges have the opposite parity for any valid edge-to-edge glueing. This is because edge-to-edge glueing is achieved by iterated zipping.

The internal edges (the hinges) are the union of those for the different glueing of the Latin cross chosen from Figure 2 so that the aim of the puzzle is to find the right subset of the internal edges as well as the right



**Figure 3:** Combined crease patterns appearing in Table 1.

matching of the boundary edges to fold the Latin cross into one of the five polyhedra. We have a design choice for the arrangement of the overlaid internal edges, as the same polyhedron can be unfolded to the Latin cross in multiple ways. For each of the four polyhedra except for the cube, an enumeration by our computer program shows that there are four distinct crease patterns, of which two are mirrors of the other two (Figure 2). For example, the tetrahedron has four crease patterns  $T_0, T_1, T_0^m$ , and  $T_1^m$ , where the superscript  $m$  indicates the mirror pattern. Therefore, we have  $4^4 = 256$  ways (128 ways up to mirror symmetry) to overlay crease patterns that can be folded into each of the five polyhedra uniquely. For example,  $Q_0T_1^mP_1O_0^m$  denotes the combination (or the overlay) of  $Q_0$ , the mirror of  $T_1$ ,  $P_1$ , and the mirror of  $O_0$  listed in Figure 2. The overlaid crease patterns for some combinations are shown in Figure 3. Now, the question is, which is the best as a puzzle? To answer this, we consider some metrics concerning the aesthetics, difficulty, ease of manufacture, and durability as in Table 1. Specifically, we consider the following six characteristics: the number of total vertices, the number of shared edges, the variance of the number of vertices in each square, the minimum angle of the panels, the number of total faces, and the minimum area of faces. We have created some promising variations and have chosen  $Q_0T_0^mP_0O_0$  in Figure 3 as the best.

## Conclusion

We have created a physical puzzle that demonstrates and materialises the wonder of a beautiful mathematical result on polyhedra. Our puzzle can be produced inexpensively with common materials. We have made the instructions downloadable at [6]. Two independent groups each consisting of five persons have tried two different combinations,  $Q_0T_0^mP_1^mO_1$  and  $Q_0T_0^mP_0O_0$ , respectively. Three participants with  $Q_0T_0^mP_0O_0$  and two with  $Q_0T_0^mP_1^mO_1$  were able to find all five polyhedra within thirty minutes. We assumed that  $Q_0T_0^mP_1^mO_1$  was much more difficult than  $Q_0T_0^mP_0O_0$  since the former has 1.5 times more vertices. However, we did not observe a significant difference in the solving time. Also, participants discovered polyhedra in a vastly different order (see [6] for the details). Every participant seemed to have enjoyed the puzzle regardless of

their mathematical backgrounds.

### Acknowledgements

We are grateful to the participants of the experiment, and R. Kawai, Y. Araki, T. Tokuda, Y. Sasaki, and S. Hisakawa for the helpful and encouraging discussions. Also, we appreciate the unknown referee’s valuable and profound comments that have substantially improved our paper.

**Table 1:** Characteristics of different combinations. The best values for each characteristics are highlighted.

combination	num. vert.	num. shared edges	var. of verts.	min. angle	num. faces	min. area
$Q_0T_0P_0O_0$	08	06	0.667	7.125	30	574
$Q_0T_1^mP_1O_0^m$	12	07	1.472	7.125	33	322
$Q_0T_1P_0O_0$	13	04	0.222	11.310	37	594
$Q_0T_1P_0^mO_0^m$	15	04	0.222	11.310	38	341
$Q_0^mT_1^mP_0^mO_1$	13	04	0.222	7.125	37	585
$Q_0T_1P_0O_0^m$	12	05	0.472	11.310	35	608
$Q_0T_1P_1O_1$	18	04	0.556	11.310	41	608
$Q_0T_1P_1^mO_0^m$	09	06	0.667	7.125	31	608
$Q_0T_1^mP_0^mO_0$	17	03	0.250	11.310	42	607
$Q_0T_0^mP_0O_0$	16	04	0.556	18.435	40	583
$Q_0T_0^mP_0O_1$	17	03	0.917	18.435	42	335
$Q_0^mT_1^mP_0^mO_1^m$	19	03	0.917	18.435	43	133
$Q_0T_0^mP_1^mO_1$	25	01	1.139	11.310	51	189

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