Polyhedra with 90° Dihedral Angles

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Abstract

The angle of 90 degrees is for sure the most used angle when two flat faces in a construction have to be connected edge to edge. Looking around in the world of building we have to admit that this angle dominates. But when we look at the way this angle is used we have to conclude that in almost any case the result is just a box, a variation of the cube, compressed or stretched in one or two directions. Is this really all that is possible? Or can the use of this connection angle of 90 degrees lead to other shapes?

Introduction

Models of the Platonic polyhedra have been made for many centuries. In 1520 Albrecht Dürer published a method for making these models from paper [1]. A model can then be made from a flat paper layout (Figure 1a,b), for which the faces are folded towards each other and then glued together edge to edge. The angle at which one face is joined to the adjacent face does not need to be calculated first with this method. We don't need to know this angle. In fact, in most cases it is not an easy matter to calculate this angle.

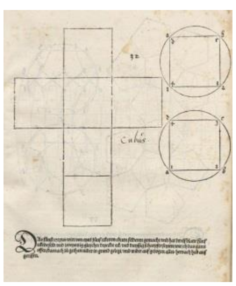

Figure 1a: *Plan for a cube.*

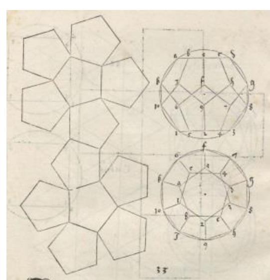

Figure 1b: *Plan for a dodecahedron.*

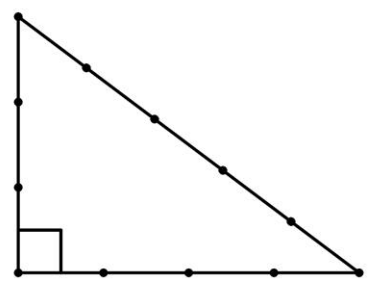

Figure 2: *3-4-5-triangle.*

Actually, normally we only know it for the cube. For the cube, this angle is 90°. A connection angle (also called a dihedral angle) of 90° is easy to make in almost any material and is therefore often used. Western architecture is pretty much characterized by it. In fact, many tools and methods have been developed to precisely construct this angle. One example is the use of the 3-4-5 Pythagorean triangle, in which one of the angles is 90 degrees (Figure 2).

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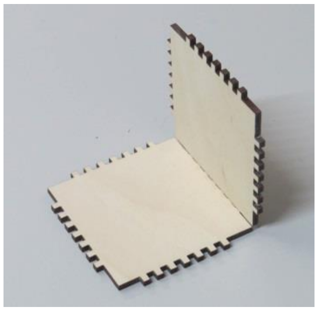
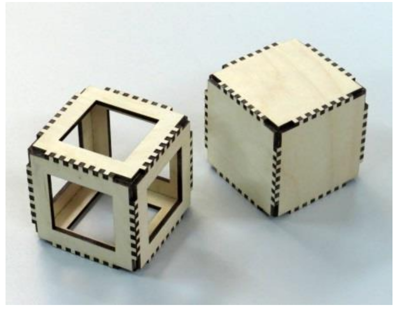
90° as a Starting Point

The question arose that if it is so pleasant to use this connection angle, why has it not led to greater freedom of design than just those rectangular box shapes? Can't there be other possibilities? Are there no more possible constructions in which all connection angles are 90°? After some searching, few real solutions emerged. But also a special one: the Jessen polyhedron (Figure 5), a variant of the icosahedron, the regular 20-faced polyhedron [2]. In this object all connection angles really are 90°.





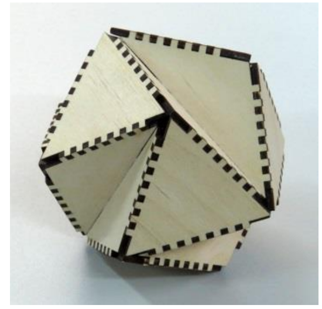
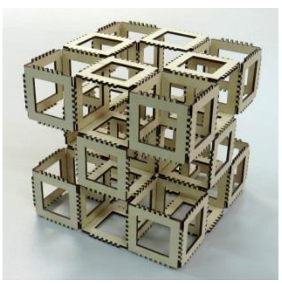


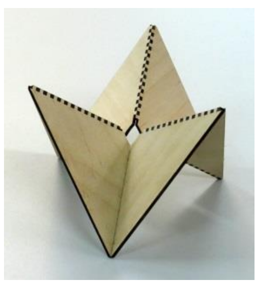
Figure 3: *Connection 90 degrees.* **Figure 4:** *Cubes.* **Figure 5:** *Jessen polyhedron.*

This has become the start of an interesting search for more possibilities. A strange search, because where should you look, and especially how should you look? It is the search for the unknown also, because I wanted to try to find solutions that are not connected to the cubic grid (the Jessen polyhedron is a good example of a solution not connected to the cubic grid).

90° Constructions from Regular Polygons

The infinitely uniform polyhedron, the Coxeter 6,6,6,6,6,6 polyhedron (Figure 6), is constructed with only 90° connections. In a corner of this polyhedron 6 faces come together as a ring. As a first step, I constructed such a ring of triangles (Figure 7). It now turned out to be possible to construct similar rings with more elements as well, in such a way that the connecting angles are exclusively 90° angles. In Figure 8 we see rings with 6, 8 and 10 elements.





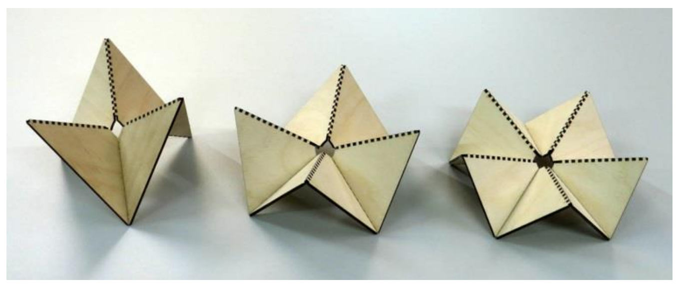
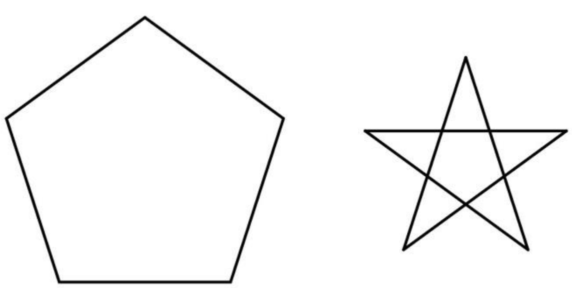


Figure 6: *Coxeter.* **Figure 7:** *Ring of 6 faces, 90 degrees connections.* **Figure 8:** *6,8 and 10 elements.*

The main figure in these constructions is a regular polygon. In Figure 8 we see 90° constructions whose upper ridges outline a triangle, a square and a pentagon, respectively. We can now turn to the more complex regular polygons, the star polygons, as possible bases for 90° constructions (Figure 9).



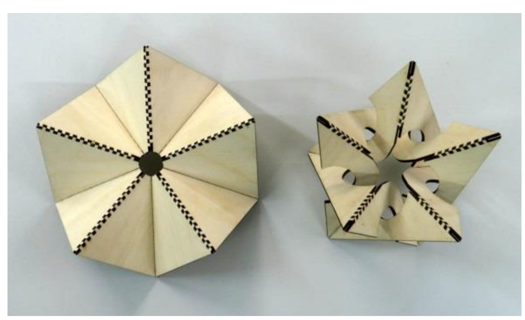


Figure 9: *Pentagon and star pentagon.* **Figure 10:** *Constructions with 90° angles.*

The five-pointed star is the first star polygon in the list, the first polygon in which the edges intersect, and in Figure 10 (right) we see the result of the 90° construction based on it. The series of star polyhedrons can be continued with the star-7-angle and the star-9-angle (Figure 11) with the resulting 90° constructions shown in Figure 12.

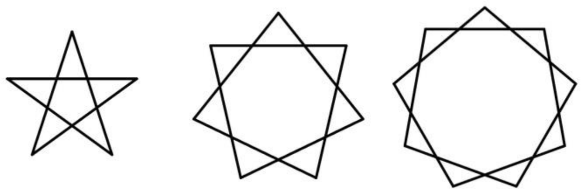


Figure 11: *Star polygons 5-2, 7-2 and 9-2.*

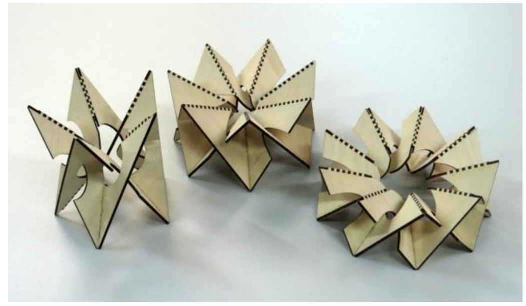


Figure 12: *Constructions with 90° angles.*

Besides the number of vertices of the polygon, we can also increase the complexity. Examples are the 90° constructions based on the 8-3 polygon (Figure 13a, b), an 8-corner polygon where we have to go around 3 times, as it were, and the 11-4 polygon, an 11-gon where we go around 4 times before we get back to the starting point (Figure 14a, b).

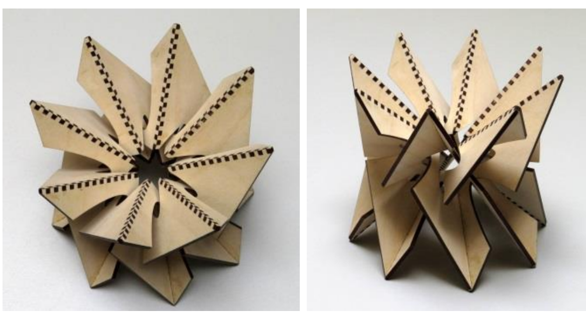


Figure 13: *Based on the star polygon 8-3.*

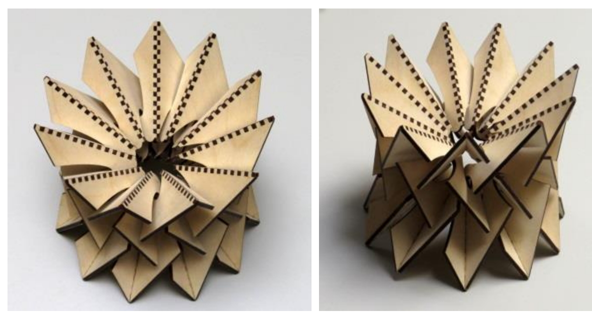


Figure 14: *Based on the star polygon 11-4.*

90° Angle Columns

Another group of forms arose when I took a central ring of the cube as a starting point. The idea is as follows: We divide the cube into three parts, cutting off two opposite corner pyramids, cutting along the diagonals of faces that meet at those corners, as shown in Figure 15.

Figure 15: *Splitting the cube.*

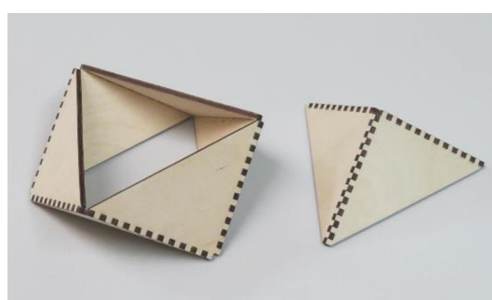


Figure 16: *Middle ring of the cube and one of the cube-pyramids.*

We then get two pyramid-shaped parts and one ring-shaped part. In Figure 16 we see the ring-shaped part and one of the pyramid-shaped parts. On the ring-shaped middle part we can now add extra faces to the open edges, the diagonal lines, at an angle of 90° (Figures 17, 18).

Figure 17: *Cube ring.*

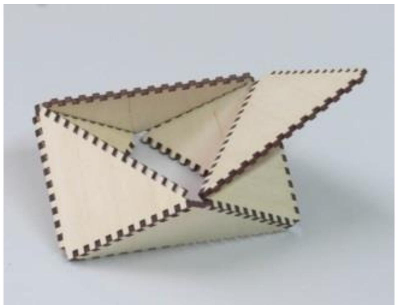


Figure 18: *One triangle added with the connecting angle of 90°.*

It now appears to be possible to make these extra faces in such a way that we can make a new 90° ring based on these new faces (Figure 19). To this end, three more triangular surfaces have been added, in such a way that only 90° connection angles are used. We can add such a new ring on both sides of the original ring to build a column. And that process can be repeated to create a series of 90° constructions, a series of columns. The resulting columns all have an odd number of rings (Figures 20, 21).

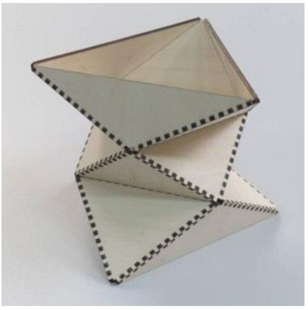


Figure 19: *Three rings.*

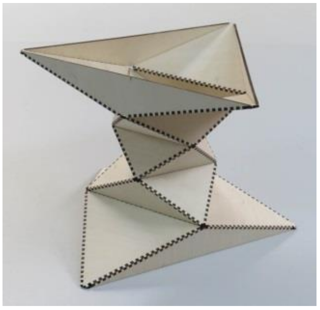


Figure 20: *Five rings.*

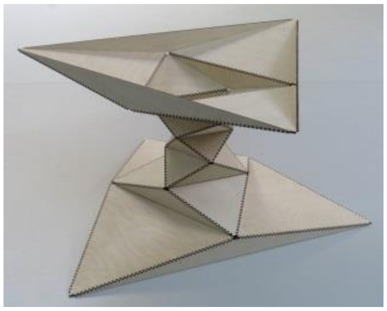


Figure 21: *Seven rings.*

In addition to this series, with the center ring of a cube as the starting figure, we can also make a series in which the starting figure is a 90° double ring (Figures 22, 23, 24). This gives us columns with an even number of rings.

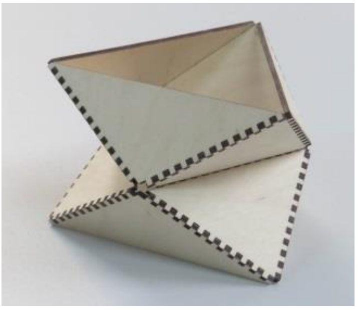


Figure 22: *Two rings.*

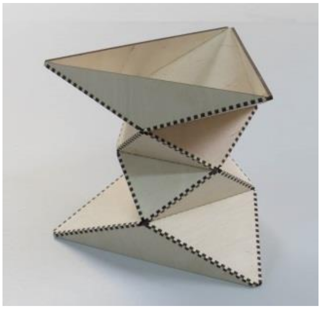


Figure 23: *Four rings.*

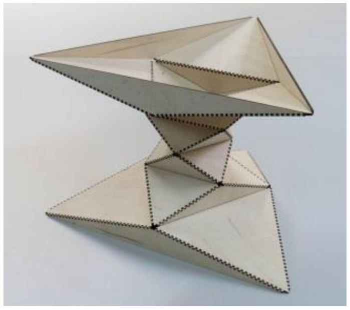


Figure 24: *Six rings.*

In Figure 25 we see the collection of the columns, from the cube ring, up to a 7-ring 90° column. Once again we see a series of constructions that we would not immediately expect when using only 90 degree connection angles in the construction.

Figure 25: *Collection of the columns from 1 to 7 rings.*

2D Infinite 90° Constructions

The pyramid-shaped parts of the cube (Figure 26) also give rise to the development of new 90° constructions. Adding a 90° ring can also be done on such a pyramid-shaped piece (Figure 27). This creates an element that we can use as a tile in a periodic pattern that extends infinitely in two directions

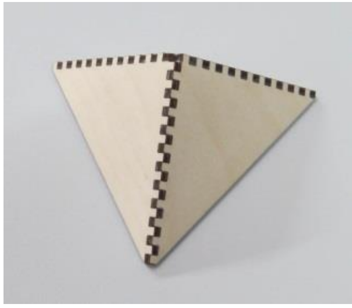


Figure 26: *Cube pyramid.*

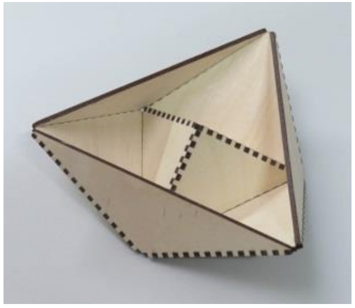


Figure 27: *Cube pyramid with one ring added.*

In the 90° construction shown in Figure 28a we see how this tile can result in a 2d infinite 90° construction. Figure 28b shows the pattern horizontally rotated 180°.

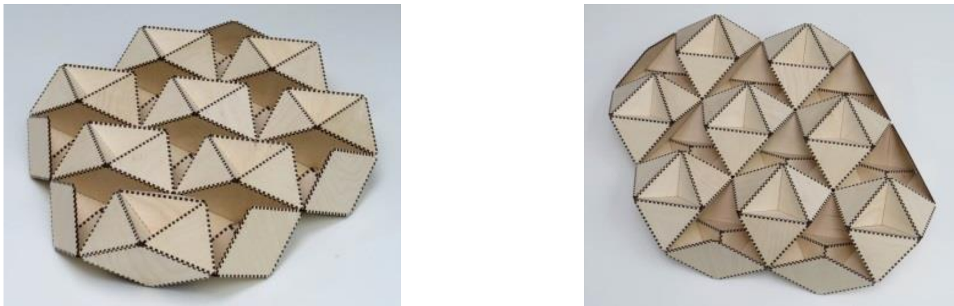


Figure 28a,b: *Two views of the 2d infinite 90° construction.*

It turns out that other divisions of a cube can also be used as a starting point for 2d infinite 90° constructions. The basic construction of Figure 29 is based on the bisecting the cube along four plane diagonals. In this case, a 90-degree ring of eight elements is constructed on it (Figure 30a,b).

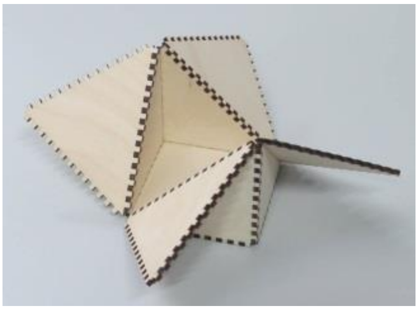


Figure 29: *Starting point.*

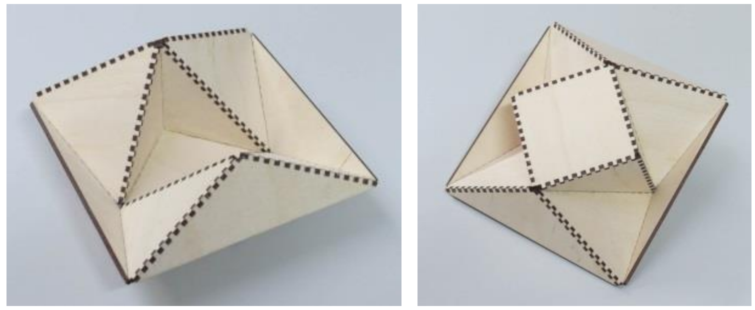


Figure 30a,b: *Tiles for 2d infinite 90°construction.*

The resulting construction (Figure 30a,b) can again serve as a tile in a 2d infinite 90°. We link this tile to the horizontally tilted version and vice versa. The result can be seen in Figure 31a,b.

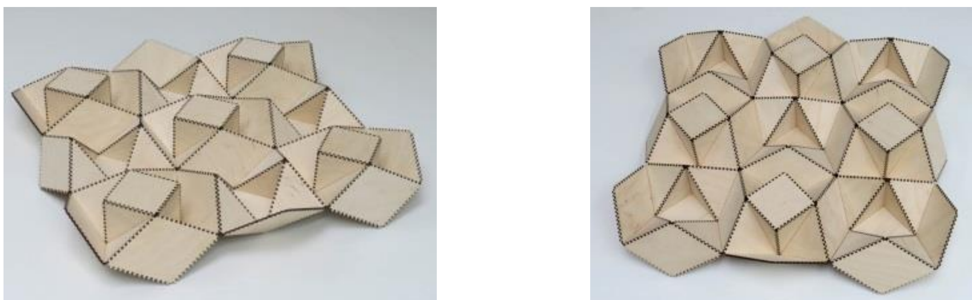


Figure 31a,b: *2d infinite 90° construction.*

A final example from this group is the tiling shown in Figure 36a,b.

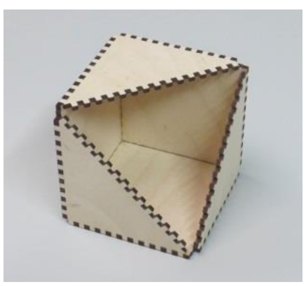


Figure 32: *Starting point.*

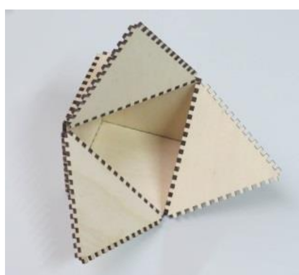


Figure 33: *Adding triangles.*

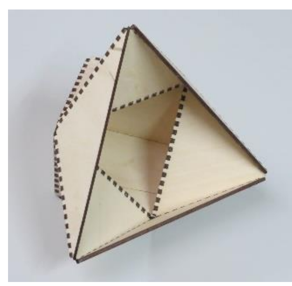
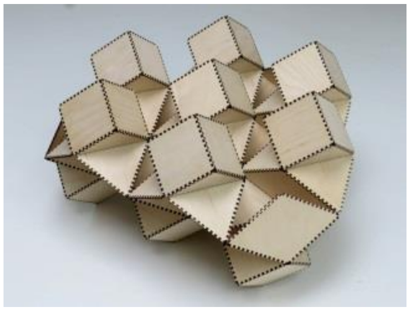


Figure 34: *Tile.*

Figure 35: *Upside down.*

The initial shape is the cube from which one pyramid corner has been removed (Figure 32). A 90° ring is constructed on top of that (Figures 33 and 34), which in turn leads to a 90° construction that can be used as a tile (Figure 35) for a 90° 2d infinite tiling (Figure 36a,b).



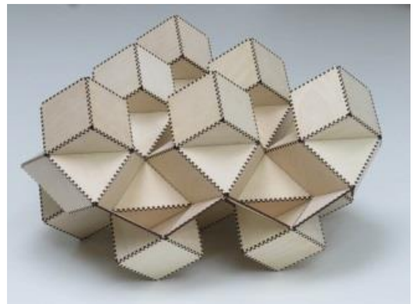


Figure 36a,b: *Another 2d infinite 90° construction.*

3D Infinite 90° Constructions

The columns shown in the paragraph '90° angle columns' can now be used again as building blocks for infinite constructions with only 90° connection angles. For example, to arrive at 2d infinite constructions, we can start with a column as in Figure 37 and cut off the top half. We then ‘close’ this half column on one side by replacing three half squares in the cube ring with three whole squares (figure 38). The result then becomes, as shown in Figure 39, a 2d infinite 90° construct.

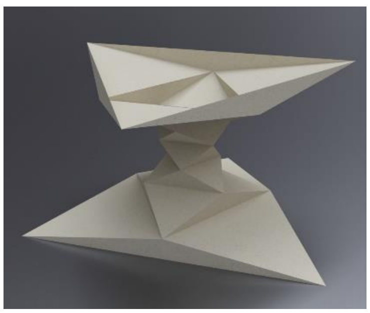


Figure 37: *Column with 7 rings.*

Figure 38: *Sealed half column.*

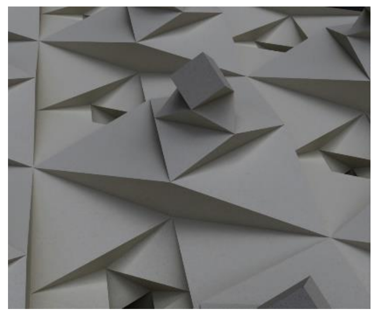


Figure 39: *2d infinite construction.*

But with the columns we can go one step further to produce 3d infinite 90° constructions. This has been worked out for the column with 7 rings (Figure 40) and for the column with 6 rings (Figure 41). We can observe the difference between the construction with the odd ring columns and the construction with even ring columns: The triangle shape in the top is twisted relative to the triangle shape in the bottom in the ‘odd version’ and is parallel in the ‘even version’.

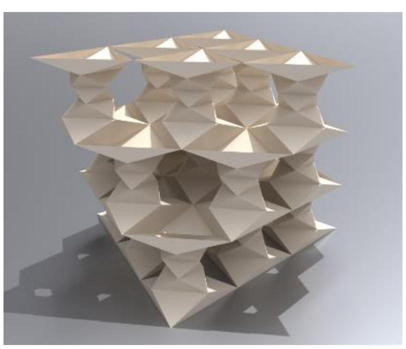


Figure 40: *3d infinite based on odd columns.*

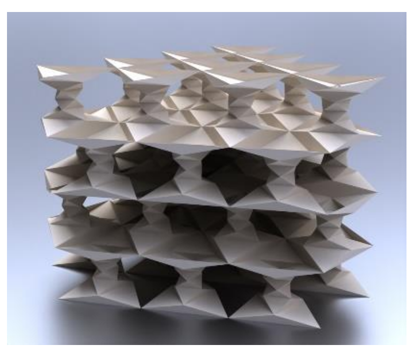


Figure 41: *3d infinite based on even columns.*

90° Dodecahedron

We started with the observation that the angles between the faces of a dodecahedron are not directly known or calculated. Is it now possible to make a dodecahedron as a 90° construction? To answer this question, let's have a look at the drawings of polyhedra by Leonardo da Vinci. In addition to drawings of the Platonic polyhedra (Figure 42), we also see an interesting variation on this: the elevation versions of the Platonic polyhedra (Figure 43). Here a pyramid is drawn on all faces of the polyhedron. We can now replace the tetrahedral-shaped pyramids in Leonardo's drawing by cube-corner pyramids, the parts of the cube that we created in Figure 16.

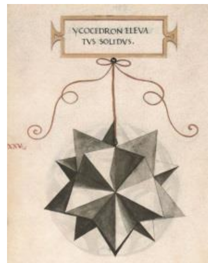


Figure 42: *Leonardo's drawing of a dodecahedron.* **Figure 43:** *Drawing of elevated icosahedron.*

On this new elevated version of the icosahedron, the angle between 2 adjacent ones is still not 90°, but when we rotate these pyramids around an axis running from the top of the pyramid to the center of the base polyhedron (Figure 44), it is possible to find a setting where this angle is exactly 90 degrees. And so eventually even a dodecahedron with only 90° angles can be realized (Figure 45).

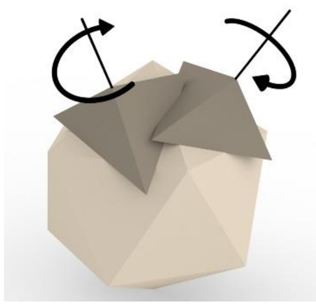


Figure 44: *Rotating the pyramids.* **Figure 45:** *Final 90 degree dodecahedron.*

Conclusion

It turns out that this group, the 90° constructs, contains many unknown members. It seems like an unexplored area, worthy of further study. I have now built about 60 90° structures, a selection of which is discussed here. And those 60 constructions seem to be just the beginning.

References

[1] A. Dürer - *Unterweisung der Messung mit dem Zirkel und Richtscheit in Linien, Ebenen und ganzen Körpern, durch Albrecht Dürer zusammengozogen und zu Nutz aller Kunstliebhabenden mit Zugehörigen Figuren in Druck gebracht im Jahr MDXXV,* Nurnberg, 1525 (2nd ed. 1538)

[2] Jessen, B. - *Orthogonal Icosahedron*. *Nordisk Mat. Tidskr*.15, 90-96, 1967