Changing Spots: Using Combinatorics to Count Japanese Braiding Patterns

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Abstract

Kumihimo is an ancient Japanese braiding technique which involves 8, 16, or more strands of fiber which are braided while being suspended from a frame or plate. The final braid can be round, flat, or in-between. Rosalie Neilson [8] explored the technique known as Kongō Gumi, which refers to a particular braiding pattern usually done with 16 strands, each chosen from 2 different colors. Neilson attempted to classify these patterns up to a natural set of symmetries. I decided to verify this count using mathematical combinatorics and discovered that Neilson was four patterns short. She and I then tracked down the missing patterns together. I also applied the technique to braids with 4 colors. The same techniques can be applied to other numbers of colors and also to other types of braids.

Kumihimo and Kongō Gumi

Kumihimo comes from the Japanese words *kumi*, coming together, and *himo*, string, cord, or rope [2]. Kumihimo is therefore just the Japanese word for braid, but there are distinctive Japanese styles of braiding that are readily recognizable. For one thing, Kumihimo is typically done with many more strands of fiber than are used in Western braiding, generally a multiple of 4 and typically 8 or more [1]. Because of the large number of strands, special equipment has been developed such as the *marudai*, a traditional stand, or the modern foam disk with notches.

Most North American Kumihimo braiders start with a style known as Kongō Gumi, the "braid as strong as metal" [7], as pictured in Figure 1a. This braid uses $n = 4k$ strands (traditionally $n = 16$) arranged into $2k$ pairs, with the pairs evenly distributed around a braiding stand or disk, as shown in Figure 2b (from underneath) and on the left side of Figure 3b (schematic, from above). Each set of opposite pairs rotates strands so that the upper-right strand becomes the new lower-right strand, and the lower-left strand becomes the new upper-left strand, as shown by the solid arrows on the left side of Figure 2a. Then the disk is rotated and the steps are repeated, as shown by the other arrows. Each strand puts color in a specific set of positions on the braid, as shown by the numbers in Figure 2a. This creates quite a bit of symmetry in the final design, shown on the right side of Figure 3b and in Figure 1b.

The group of symmetries of an n-strand Kongō Gumi braid is a three-dimensional line group, which can be thought of as a wallpaper group wrapped around a cylinder. If every strand is the same color and n is divisible by 4, the group is isomorphic to a dihedral group and is generated by n helical rotations around the braid's axis and a 180° rotation around a line perpendicular to the axis. ($\mathrm{P}n_{n/4}22$ in Hermann-Mauguin crystallographic notation [9, Sec. 10.1].) Kongō Gumi is chiral, so there are no reflection symmetries.

We will consider two braid patterns equivalent if one can be changed into the other using these symmetries. We also wish to consider two multicolored braids equivalent if one can be changed into the other by simply changing which color is which. This puts us into the territory of de Bruijn's generalization [4] of the Pólya enumeration theorem from combinatorics.

(a)

(b)

Figure 1: *(a) 16-strand Kongō Gumi braid, c. 10 in. length not including ring. (b) Detail from Figure 1a, c. 1 in. length.*

De Bruijn's Theorem

De Bruijn's Theorem works by associating every symmetry of an object with its set of cycles. Suppose an object has a group of symmetries G of size k, and there are m locations on the object which are permuted by the symmetries. For example, in a 16-strand Kongō Gumi braid, $k = 32$ symmetries and $m = 16$ locations. Given a symmetry, a cycle of that symmetry is a set of locations which can be reached from each other by using that symmetry. For example, the cycles of the 90° rotation of a 16-strand Kongō Gumi braid around its longitudinal axis are shown in Figure 3a. A complete analysis appears in Supplement 1. (For more background on cycles, see Riordan [10, Chap. 5].)

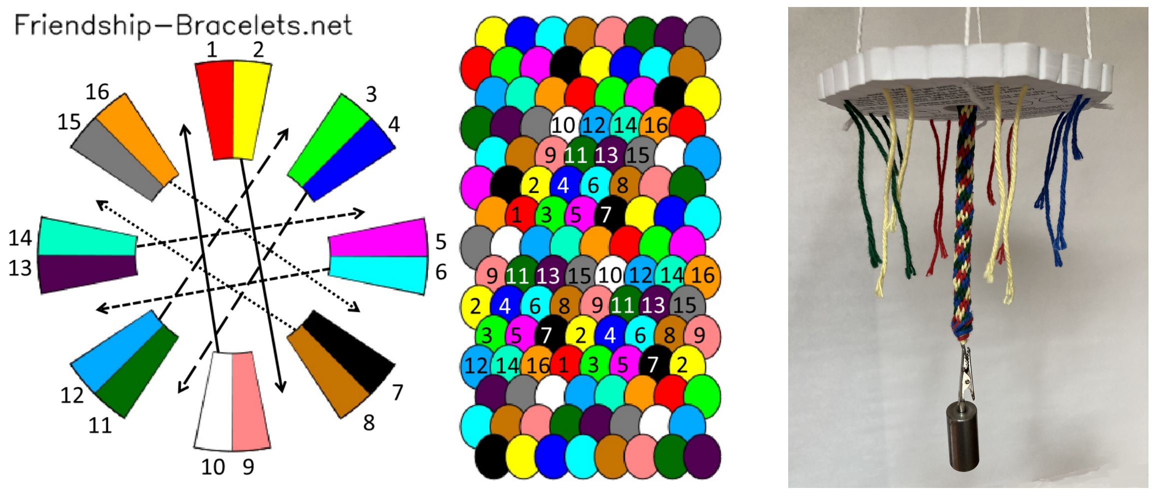
De Bruijn's Theorem [5] now tells us the number of non-equivalent ways to color an object with a specific group of symmetries and a set of locations which can each be colored with one of q different colors. Applying this theorem to 16-strand dihedral symmetry and 2 colors, we can calculate that there are 1162 non-equivalent Kongō Gumi braid patterns, including the one which only uses one color. Similarly, there are 5607437 non-equivalent 16-strand Kongō Gumi patterns which use 4 or fewer colors, with 5380907 having exactly four colors, 225368 having exactly three colors, and 1162 having one or two colors as before.

Comparing with Neilson

Rosalie Neilson's book [8] attempted to produce a complete catalog of the non-equivalent 16-strand Kongō Gumi patterns of exactly 2 colors, classified according to how many "spots" the pattern had, or equivalently, how many strands were colored in each color. (For example, Figure 3b is an 8-spot braid, since it can be seen as 8 blue spots on a purple background.) Since switching the colors gives an equivalent braid, only the numbers of spots from 1 to 8 needed to be considered. (Figure 3b is also 8 purple spots on a blue background.)

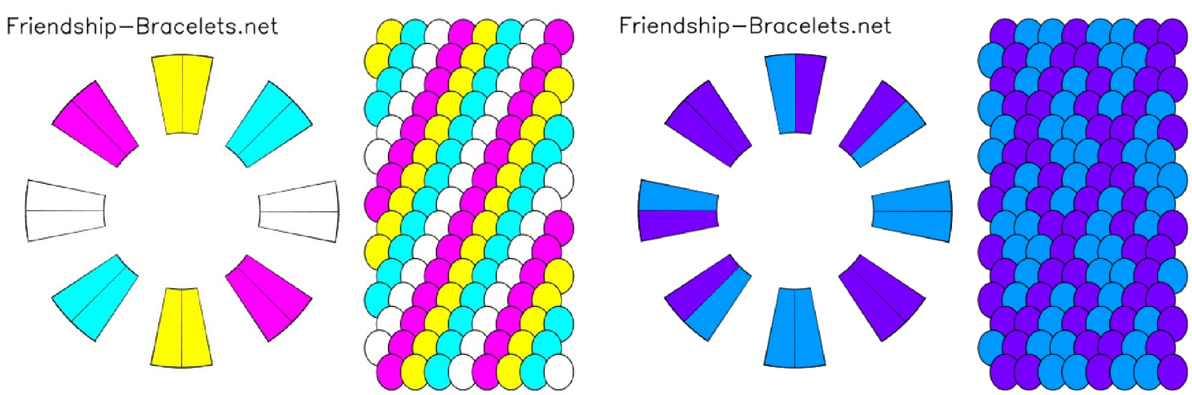
The book builds up the patterns recursively, starting with one spot and gradually adding spots to the pattern. However, the book only catalogs 1157 patterns with exactly 2 colors, whereas my calculations predicted 1161. In order to find the missing braids, I used a more complete version of de Bruijn's Theorem [4, Thm. 1] which allows us to weight the patterns according to the number of strands of each color.

This allowed me to compile Table 1 and determine that Neilson was missing one 4-spot pattern and three 8-spot patterns. In order to find the missing patterns I turned to the Combinatorial Object Server (COS) [3], which let me generate strings corresponding to 2-color, 16-strand patterns with a fixed number of spots, with equivalence up to rotation but not up to color exchange. For the 4-spot patterns the color exchange does not



(a) (b)

Figure 2: *(a) Making the Kongō Gumi braid and the correspondence between strands on the disk and colors in the pattern. (b) A braid in progress, hanging from foam disk. White strings for presentation only.*



(a) (b)

Figure 3: *(a) The cycles of the 90° rotation of the 16-strand braid around its longitudinal axis. Each color represents a set of locations mutually reachable by 90° rotation. (b) The Kongō Gumi pattern used in Figure 1a. Pattern by Rosalie Neilson. Both pictures generated by the author using Friendship-Bracelets.net.*

Spots:	1	2	3	4	5	6	7	8	Total
Patterns:	1	8	21	72	147	280	375	257	1161

Table 1: *Inventory of patterns given by de Bruijn's Theorem.*

affect the results, and I could use a small amount of Maple code and a spreadsheet to resort the COS output according to Neilson's scheme and find the missing pattern (1.2//6.14/ in Neilson's notation). For the 8-spot patterns the Maple code needed to be slightly more extensive in order to find the pairs which were equivalent under color exchange, but the same basic procedure revealed that the missing braids were 1.2.9/3.12/6/7.15, 1.2.9/3/6.13.14/15, and 1.2.9/4/6.13.14/16 in Neilson's notation. Diagrams for these are in Supplement 2.

Similarly, we can calculate the number of non-equivalent Kongō Gumi patterns of any specified distribution of strands across any number of colors. For example, there are 83488 non-equivalent 16-strand patterns with four strands of each color, 788865 with 3 of one color, 4 each of two colors, and 5 of the fourth color, and so on. The complete list may be found in Supplement 3.

The calculations above partially overlapped with those of Gilbert and Riordan [5], who used the simpler version of de Bruijn's Theorem, and those of Hoskins and Street [6], who used a different method. Interestingly, the goal of Hoskins and Street was to solve a different problem from fiber arts, that of cataloging the number of non-equivalent simple twills woven on a loom with n harnesses.

Summary and Conclusions

The same techniques can be applied to Kongō Gumi braids of other sizes and color distributions, other round braids such as Edo Yatsu Gumi, and possibly other shapes such as square braids and flat braids. Neilson has already published (much smaller) catalogs of 8-strand Hira Kara Gumi and 8-strand Edo Yatsu Gumi. These could perhaps be extended to more strands using the methods of this paper, and new catalogs could be constructed of the many other sorts of traditional and modern braids.

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