A Fish Pattern on a Regular Triply Periodic Polyhedron

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Abstract

We have designed a pattern of Escher-inspired fish that tessellates the infinite regular {6,6|3} triply periodic polyhedron. We will discuss some of the background and the motivation for this pattern.

Introduction

The aim of this project was to design an Escher-inspired fish pattern that tessellates the infinite regular triply periodic polyhedron {6,6|3}. Figure 1 shows the resulting pattern printed on card stock.

Figure 1: *Part of the* {6,6|3} *polyhedron with an Escher-inspired fish pattern.*

In 2021 [2], we used papercrafting technology to create a similar fish pattern on the {4,6|4} polyhedron, shown in Figure 2 below. This pattern solved all but one of the problems we saw in our 2012 fish pattern on that polyhedron which is shown in Figure 3 below. Solving that final problem was the goal of this project.

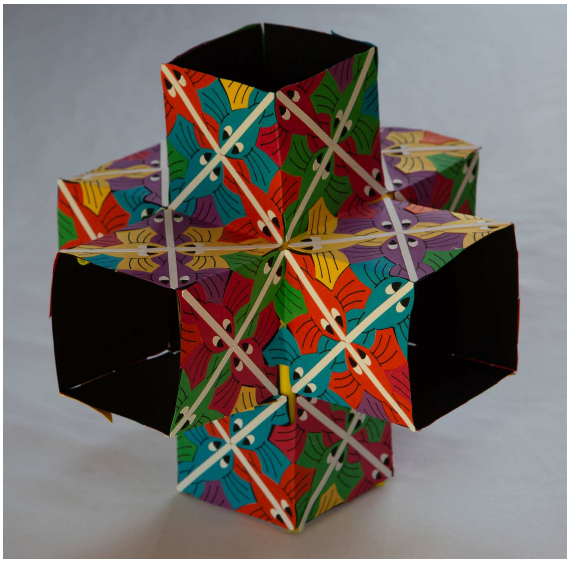

Figure 2: *The 2021 fish pattern on the* {4,6|4} .

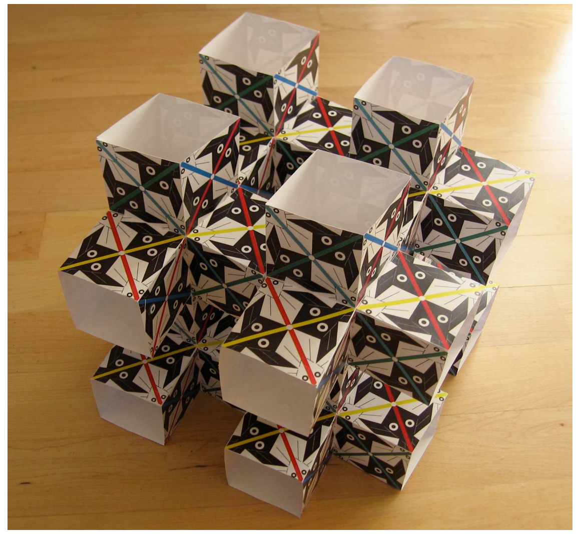

Figure 3: *The 2012 fish pattern on the* {4,6|4} .

In the next section, we discuss regular triply periodic polyhedra. Then we talk about the aesthetic problems with the patterns of Figures 2 and 3. We note that M.C. Escher had similar problems with his hyperbolic pattern *Circle Limit I*. He was able to fix them in his woodcut *Circle Limit III*. Next we explain how the polyhedron of Figure 1 overcame the final flaw in the polyhedron of Figure 2. Then we discuss the parallel properties of the backbone lines of Figure 1. Finally we indicate possible directions for future work.

Triply Periodic Polyhedra and the Previous Polyhedron

People have used regular tessellations for artistic purposes for millennia. Those tessellations are composed of regular p-sided polygons, or *p-gons*, that meet q at each vertex, and are denoted by the Schläfli symbol $\{p, q\}$. One can consider regular tessellations in each of the three "classical" geometries: spherical geometry, Euclidean geometry, and hyperbolic geometry. The tessellation $\{p, q\}$ is spherical, Euclidean, or hyperbolic depending on whether $(p-2)(q-2)$ is less than, equal to, or greater than 4, respectively. There are the five regular spherical (Platonic) tessellations: $\{3, 3\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 3\}$, and $\{5, 3\}$. There are also the three regular Euclidean tessellations: $\{3, 6\}$, $\{4, 4\}$, and $\{6, 3\}$, and an infinite number of regular hyperbolic tessellations $\{p, q\}$ with $(p-2)(q-2) > 4$.

The curvature of regular polyhedra occurs only at their vertices, so we can consider polyhedra to be hyperbolic if the vertex angle sums are all greater than 2π. There exist a number of such polyhedra that are composed entirely of p-gons meeting q at a vertex (so $(p-2)(q-2) > 4$), which are also denoted $\{p, q\}$, by extension of the Schläfli symbol. If one of those polyhedra repeats in three independent directions in Euclidean 3-space, it is called *triply periodic*. If its symmetry group is also transitive on vertices, edges, and faces, it is considered to be *regular*. H.S.M. Coxeter called these *regular skew polyhedra*. In 1926 he and John Flinders Petrie proved that there are exactly three of these: {4,6|4}, {6,4|4}, and {6,6|3}, where {p,q|r} also extends the Schläfli symbol, denoting a polyhedron composed of p-gons meeting q at each vertex, with regular r-sided polygonal holes [1, 7]. These polyhedra are considered to be hyperbolic analogs of the Platonic solids and the regular Euclidean tessellations. In addition to the patterns on the {4,6|4} and {6,6|3}, we also designed patterns on the {6,4|4} polyhedron [4]. The {4,6|4} polyhedron may be the easiest to understand since it is based on the cubical tessellation of Euclidean 3-space. The {6,6|3} polyhedron can be considered to be composed of (invisible) tetrahedron "*hubs*" connected by truncated tetrahedron "*struts*" attached to each of the hub faces by their (missing) triangular faces, leaving only the regular hexagonal faces of the truncated tetrahedra intact, as seen in Figure 1.

Problems with Escher's *Circle Limit I* and the Figure 3 Polyhedron

In 1958 Escher created his first "hyperbolic" pattern *Circle Limit I*, which can be considered to be a tessellation of the hyperbolic plane by angular black and white fish [5]. Our rendition of that pattern is shown in Figure 4. Though it was his first "circle limit" pattern, Escher was dissatisfied with it because it had these three shortcomings as he saw it:

1. The fish were not consistently colored along backbone lines — they alternated from black to white and back every two fish lengths.
2. The fish also changed direction every two fish lengths — thus there was no "traffic flow" (Escher's words) in a single direction along the backbone lines.
3. The fish are very angular and not "fish-like" (we only know they are fish because Escher said they were).

In 1959 Escher solved each of these problems in his pleasing woodcut *Circle Limit III* [6]. Our rendition is shown in Figure 5. The fish are all the same color along a backbone line, they swim head to tail, and they are quite fish-shaped. One sees that the pattern of Figure 2 resolves Escher's first and third criticisms of the

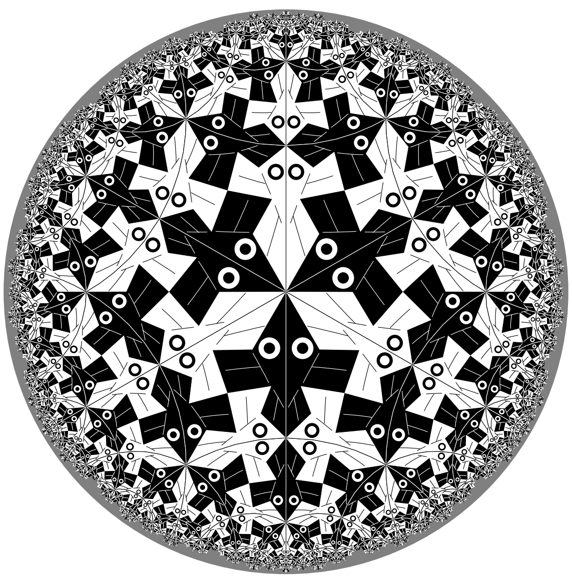

Figure 4: *A rendition of* Circle Limit I.

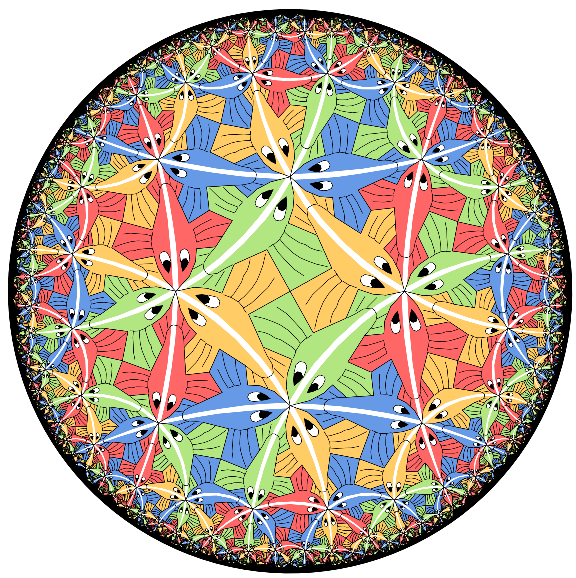

Figure 5: *A rendition of* Circle Limit III.

pattern in Figure 3. It also resolves another problem that exists in Figure 3: the lines of fish of one color are not all parallel; the yellow lines on top are at a 90-degree angle to the yellow lines underneath (and the same is true for the other colors), as explained in [2].

However the Figure 2 pattern does suffer from Escher's second concern: the fish alternate directions every two fish-lengths along backbone lines. To solve the second problem on a {p,q|r} polyhedron, both *p* and *q* need to be divisible by an odd number in order that all the fish swim head to tail, so the only possibility is {6,6|3}, which was the inspiration for the pattern of Figure 1.

Parallel Properties of the Backbone Lines of the Colored Fish of Figure 1

One of the goals achieved by the pattern on the Figure 1 polyhedron was to have all the fish along a backbone line be the same color. There are six sets of parallel backbone lines embedded in the {6,6|3} polyhedron that pass through the centers of the hexagons. Those six sets are parallel to the edges of the regular tetrahedra that comprise the hubs of the {6,6|3} polyhedron (all the hubs have the same orientation since they are all just translations of each other; and the same is true of the struts). Thus the fish along parallel backbone lines can be colored by one of six colors: red, orange, yellow, green, blue, and purple.

Also if a hexagonal face has complete fish of three colors, the fins that overlap onto that face have complementary colors. For example if the complete fish are colored green, yellow, and orange, the overlapping fins are colored red, purple, and blue. Thus all six colors appear on each face.

In addition to having fish of three colors along backbone lines through the centers of the faces, three backbone lines of different colors pass through the vertices where six hexagons meet. This is shown in Figure 7 of [4].

There is a subtlety in this pattern in that in the parallel lines of one color, the fish in half the lines are swimming one direction and the fish in the other half of the lines are swimming in the opposite direction. This can be seen in the yellow fish in Figure 1: they are swimming left-to-right on the upper/horizontal hexagons and right-to-left on the downward slanted hexagons toward the front.

Conclusions and Future Work

We have shown a new pattern on the {6,6|3} polyhedron as shown in Figure 1 that fixes all of the problems of the polyhedron pattern in Figure 3, including Escher's "traffic flow" problem.

We think that this would be a good candidate for a polyhedral pattern to be rendered using craft technologies such as papercrafting or embroidery.

Another direction would be to render this pattern on Schwarz' D-surface, a triply periodic minimal surface that has the same topology as the {6,6|3} polyhedron.

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