Self-Similar Quadrilateral Tilings and Deployable Scissor Grids

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Abstract

From a self-similar quadrilateral tiling, we construct a scissor grid by replacing each quadrilateral with a scissor linkage. We show that the resulting linkage is deployable if and only if the quadrilaterals are cyclic or parallelograms.

Introduction

Starting with any convex quadrilateral Q, a tiling can be made by translating, rotating, and scaling copies of Q so that they match along edges. See Figure 1. We call such a tiling a *self-similar quadrilateral tiling*. An image of such a tiling appears as Figure 3.17 of Thurston's book, *Three-Dimensional Geometry and Topology* [2]. Unless Q is a parallelogram, the tiling has a limit point at which the sizes of the tiles approaches zero. Locally, the tiling is planar, but it does not link up with itself as it wraps around its limit point. Formally, we think of it as a tiling of the universal cover of the plane punctured at the limit point.

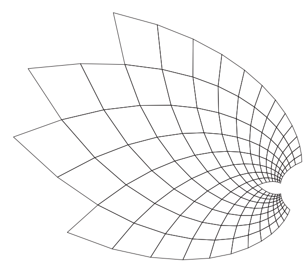


Figure 1: *Part of a self-similar quadrilateral tiling.*

Scissors are formed from two rigid arms attached to each other at a pivot. By attaching the ends (labelled A, B, C, and D in Figure 2) of scissors to each other with further pivots we may construct larger linkages. A linkage is *deployable* if it can change shape. Such linkages have a long history in kinematic sculpture and deployable architecture [1].

We restrict to *grids* of scissors $\{S_{i,j} \mid i, j \in \mathbb{Z}\}$, where ends A and D of scissor $S_{i,j}$ are attached at pivots to ends B and C (respectively) of $S_{i+1,j}$. Similarly, A and B of $S_{i,j}$ are attached to D and C (respectively) of $S_{i,j+1}$. Each scissor S determines a quadrilateral Q_S (given an angle between its arms), by taking the two arms as the diagonals of Q_S. We require that the quadrilaterals of neighboring scissors are coplanar and intersect only along a shared edge. Thus a grid of scissors G determines a quadrilateral tiling Q_G.

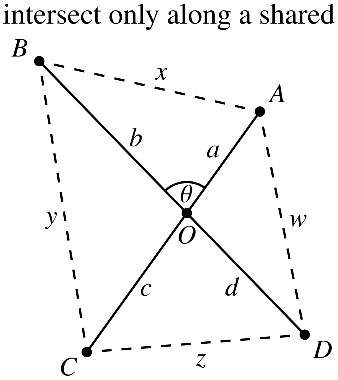


Figure 2: *A scissor at a given angle θ.*

A scissor grid may or may not be deployable: as a scissor's arms rotate, it induces motion in its neighbors. These propagating motions must be consistent to allow a global movement. In this paper, we prove the following:

Theorem. *Let G be a scissor grid. The following are equivalent:*

(i) For some configuration of G, the quadrilaterals of Q_G are similar, and G is deployable.

(ii) For some configuration of G, the quadrilaterals of Q_G are similar and are cyclic or parallelograms.

(iii) For any configuration of G, the quadrilaterals of Q_G are similar and are cyclic or parallelograms, and G is deployable.

*This research was partially supported by the Koslow Undergraduate Mathematics Research Experience Scholarship. We thank Saul Schleimer for helpful conversations.

A *cyclic* quadrilateral is one whose vertices lie on a circle. In order to prove this result, we first set up some notation. Consider an arbitrary scissor as shown in Figure 2. The arms AC and BD are connected at pivot O. Let a, b, c, and d be the *scissor arm lengths*: the distances from the pivot to the ends of the scissor arms. Let θ be the angle between the two arms of the scissor; this changes as they rotate against each other. Let x, y, z, and w be the distances between endpoints of the scissor (the side lengths of the quadrilateral); these depend on θ. The scaling factor between adjacent quadrilaterals in a self-similar tiling is defined by the ratio of opposite sides of each quadrilateral Q. That is, the tile above the quadrilateral shown in Figure 2 is scaled by a factor of $\mu = \frac{x}{z}$. The tile to the right is scaled by a factor of $\lambda = \frac{w}{y}$.

Proof of the Theorem

We first show that (i) $\implies$ (ii). Let G be a deployable scissor grid, and assume that the quadrilateral tiling Q_G is self-similar in at least one configuration. Consider the ith column of scissors $\{S_j = S_{i,j} \mid j \in \mathbb{Z}\}$ in G. We use notation as in Figure 2 with corresponding indices. Since G is self-similar in at least one configuration, we have that for some fixed scaling factor $\mu = \frac{x_0}{z_0} = \frac{x_j}{z_j}$, neighboring scissor arm lengths are related by μ. That is, $a_{j+1} = \mu a_j$, $b_{j+1} = \mu b_j$, and so on. Let $t_j = \cos\theta_j$. Combining the law of cosines applied to the triangles $A_jO_jB_j$ and $C_{j+1}O_{j+1}D_{j+1}$, we get the following recurrence relation:

$$t_{j+1} = \frac{a_j b_j}{c_{j+1}d_{j+1}} t_j + \frac{c_{j+1}^2 + d_{j+1}^2 - a_j^2 - b_j^2}{2c_{j+1}d_{j+1}}. \tag{1}$$

Writing $a_0 = a, b_0 = b$, and so on, we can rewrite the recurrence relation as follows:

$$t_{j+1} = \frac{ab}{\mu^2 cd} t_j + \frac{\mu^2c^2 + \mu^2d^2 - a^2 - b^2}{2\mu^2 cd}. \tag{2}$$

Setting $\alpha = \frac{ab}{\mu^2 cd}$ and $\beta = \frac{\mu^2c^2+\mu^2d^2-a^2-b^2}{2\mu^2cd}$, we arrive at the first order linear recurrence relation $t_{j+1} = \alpha t_j + \beta$. Standard techniques for solving linear recurrence relations give the following solutions. If $\alpha \neq 1$ then $t_j = \left(t_0 - \frac{\beta}{1-\alpha}\right)\alpha^j + \frac{\beta}{1-\alpha}$. If $\alpha = 1$ then $t_j = t_0 + j\beta$. Since $t_j = \cos\theta_j$, we have that $t_j \in [-1, 1]$. If $\alpha \neq 1$ then for sufficiently positive ($\alpha > 1$) or negative ($\alpha < 1$) indices j, the value of t_j will exit $[-1, 1]$, unless $t_0 = \frac{\beta}{1-\alpha}$. In this case, there is only one possibility for t_0, which contradicts the fact that G is deployable. If $\alpha = 1$ then for sufficiently large j, t_j again exits $[-1, 1]$ unless $\beta = 0$. Eliminating μ from the two equations $\alpha = 1$ and $\beta = 0$, we get an expression relating the arm lengths. Applying the same argument to an infinite row of scissors gives the same equation, but with a and d playing the role of a and b in the column of scissors. We get:

$$\frac{ab}{cd} = \frac{a^2+b^2}{c^2+d^2} \quad \text{and} \quad \frac{ad}{cb} = \frac{a^2+d^2}{c^2+b^2}. \tag{3}$$

Cross multiplying, we see that $(cd)a^2 - (c^2+d^2)ab + (cd)b^2 = 0$ and $(cb)a^2 - (c^2+b^2)ad + (cb)d^2 = 0$. These can be factored as $(ac - bd)(ad - bc) = 0$ and $(ac - bd)(ab - cd) = 0$. Therefore:

$$(ac = bd \quad \text{or} \quad ad = bc) \quad \text{and} \quad (ac = bd \quad \text{or} \quad ab = cd). \tag{4}$$

The first possibility from each pair of criteria is a defining characteristic of a cyclic quadrilateral. The second criterion from each pair gives that $a = c$ and $b = d$, so we have the diagonals of a parallelogram. Using the configuration given to us by (i), the similarity condition is immediate and we have obtained (ii).

We now prove (ii) $\implies$ (iii). Consider a scissor S in G in the given configuration. The conditions of Q_S being cyclic or a parallelogram do not depend on the angle θ, so we need only show that G is deployable and that the scissors remain similar for any configuration of G. Since the quadrilateral Q_S is cyclic or is a parallelogram, we have Equation (4). Following the algebra backwards, we obtain Equation (3). Rearranging these and choosing an arbitrary angle θ, we have

$$(a^2+b^2)cd(-2\cos\theta) = (c^2+d^2)ab(-2\cos\theta) \quad \text{and} \quad (a^2+d^2)cb(2\cos\theta) = (c^2+b^2)ad(2\cos\theta) \qquad (5)$$

Setting $u = a^2+b^2$, $v = c^2+d^2$, $r = -2ab\cos\theta$, and $s = -2cd\cos\theta$, the first equation here gives $us = vr$. So $uv + us = uv + vr$, which gives us that $\frac{u}{v} = \frac{u+r}{v+s}$. Writing this in terms of a, b, c, d, and θ again, and performing a similar calculation for the second equation, from the law of cosines we obtain:

$$\frac{a^2+b^2}{c^2+d^2} = \frac{a^2+b^2-2ab\cos\theta}{c^2+d^2-2cd\cos\theta} = \frac{x^2}{z^2} \quad \text{and} \quad \frac{a^2+d^2}{c^2+b^2} = \frac{a^2+d^2+2ad\cos\theta}{c^2+b^2+2cb\cos\theta} = \frac{w^2}{y^2}. \qquad (6)$$

Therefore x^2/z^2 and w^2/y^2 are independent of θ (and are equal to μ^2 and λ^2 respectively). In the given self-similar configuration of G, any scissor has arm lengths scaled by some fixed factor compared to S. Inspecting Equation (6), we then see that μ^2 and λ^2 do not depend on the choice of scissor in G. Let S_1 be the neighboring scissor above S in G. In any configuration of G, we must have that $x = z_1$. The law of cosines applied to triangles AOB and $A_1O_1B_1$ then gives:

$$\cos\theta = \frac{a^2+b^2-x^2}{2ab} = \frac{a^2+b^2-z_1^2}{2ab} = \frac{\left(a^2+b^2-z_1^2\right)\mu^2}{2ab\mu^2} = \frac{a_1^2+b_1^2-x_1^2}{2a_1b_1} = \cos\theta_1. \qquad (7)$$

Therefore S and S_1 fit together properly if and only if $\theta = \theta_1$. (The possibility that $\theta = -\theta_1$ is ruled out by the assumption that the quadrilaterals for neighboring scissors do not overlap.) Similar arguments show that any pair of neighboring scissors in G fit together if and only if they have the same angle. Thus we have a global configuration of G if and only if all scissors have the same angle, if and only if all quadrilaterals of Q_G are similar. Thus G is deployable and we have (iii). The implication (iii) $\implies$ (i) is trivial. □

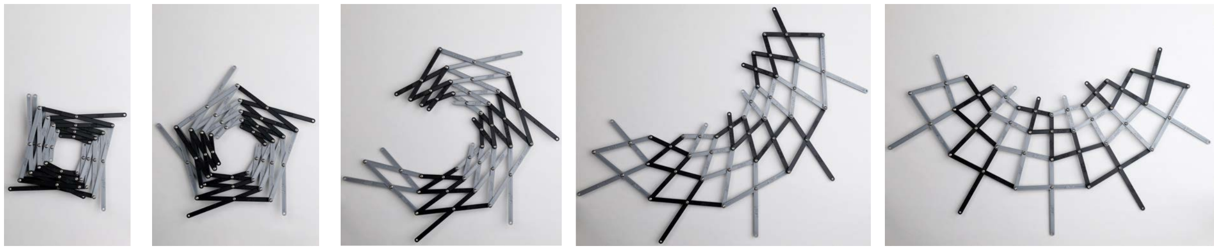
Construction

The parallelogram case is unsurprising, but the cyclic case is more interesting. To demonstrate the motion accompanying such grids, we constructed finite sections of self-similar scissor grids, shown in Figures 3, 4, and 5. A video of these is available at https://youtu.be/jjUpJCTPXaM. We 3D printed the scissor arms and connected them with bolts and lock nuts. The linkage's range of motion is limited by self collision only: if the links were allowed to pass through each other and their pivots, the mechanism would freely rotate. The links are stacked at varying heights, allowing them to pass over each other, as seen in Figures 4 and 5.

(a) **(b)** **(c)** **(d)** **(e)**

Figure 3: *A grid based on a scissor with arm lengths proportional to* $a = 12$, $b = 8$, $c = 6$, $d = 9$.

The grid in Figure 3 has arm lengths for each scissor in simple integer ratios that satisfy the cyclic quadrilateral condition. These give $\lambda = \frac{w}{y} = \frac{3}{2}$ and $\mu = \frac{x}{z} = \frac{4}{3}$.



(a) **(b)** **(c)** **(d)** **(e)**

Figure 4: *A grid with $\lambda = \mu$.*

Figure 4 shows a grid with $\lambda = \mu$. In this case, from Equations (3) and (6) we get that for each scissor, $\frac{ab}{cd} = \frac{ad}{cb}$, and so $b = d$. This means that when $\theta = \pi/2$ (as seen in Figure 4e) the grid has a mirror symmetry. Continuing the movement beyond Figure 4e, the grid collapses in a mirrored fashion, except that the black and grey columns spiral around the final square configuration. In Figures 4a and 4b we see a new phenomenon: the grid lines up with itself and gains a rotational symmetry. This is made possible by the fact that there are scissors that are not only similar but also congruent to other scissors in the grid. Since $\lambda = \mu$, the scissors of a fixed size lie along a $(1, -1)$ diagonal in the grid. Because these all have the same size they must necessarily be related to each other by a rotation around the limit point of the self-similar quadrilateral tiling. In certain configurations, these rotations line up perfectly.

(a) **(b)** **(c)** **(d)** **(e)** **(f)** **(g)** **(h)**

Figure 5: *A grid with $\lambda = \mu^2$.*

In Figure 5 the two collapsed states are different from each other. Scissors along a $(1, -2)$ diagonal are congruent, since $\lambda = \mu^2$. Thus, this grid can also line up with itself and have a (finite) rotational symmetry.

Future Directions

We would like to understand deployable grids which are not made from self-similar scissors. We expect that more general constructions of this kind would allow a designer to produce scissor grids with bespoke shape-shifting properties. In another direction, a tiling of cyclic quadrilaterals induces a grid of circles. Is there a relation between these grids and circle packings?

References

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