Nice Knots

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Abstract

When drawing knots, practitioners tend to present their simplest projections. This paper investigates what constitutes a "nice" knot drawing and how this may be measured, and describes a simple method for identifying which face in a knot's Gauss code to choose as perimeter in order to yield a "nice" drawing.

Drawing Knots

Figure 1 shows two knot drawings with three crossings each. You are probably familiar with the drawing on the left but do you recognise the drawing on the right?

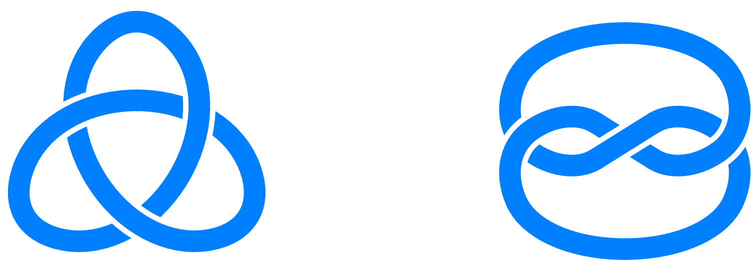

Figure 1: *Do you recognise these two knot drawings?*

These are in fact both 2D projections of the same knot (denoted 3_1), but the first design – the well-known *trefoil* – is used almost universally while the second design is rarely used. Why is this?

The short explanation is that the first projection is visually simpler. This appears to be an unwritten rule of thumb applied throughout the knot theory literature, with the standard knot tables typically showing the simplest form of each knot (with some exceptions). However, I could find no definitive discussion of this practice in the literature, and knot theorist Colin Adams confirms that there is no convention on this point.[1]

Nice Knot Drawings

Some knot drawings may be described as being "nicer" than their alternatives, where "nice" in this context refers to the simplicity and comprehensibility of the drawing rather than its aesthetic or artistic qualities. A practical test of a knot drawing's comprehensibility is to study the drawing for a few seconds and then try to redraw the knot from memory. For example, the familiar trefoil projection of 3_1 (Figure 1, left) is easy to draw from memory, while the less familiar projection (right) can be surprisingly difficult. This effect quickly scales up with the number of crossings.

Software packages for knot generation typically allow the user to choose a preferred projection from the available choices [2]. Rideout goes a step further to recommend that the chosen projection's perimeter face should contain: *"a large number of edges"* [6]. I believe that the case can be stated more strongly and that there exist clear and simple guidelines for identifying knot projections more likely to be preferable to others.

[1] Personal correspondence, 14/1/2022.

Perimeter Choice

A convenient way to describe knots is by their *Gauss code*, which lists the crossings in the order in which the knot path visits them, with underpasses negated [5]. For example, the Gauss code of the 3_1 knot shown in Figure 1 is $g = \{1, -2, 3, -1, 2, -3\}$. This can then be used to generate an *arc diagram* of the knot consisting of a *vertex* for each crossing and an *edge* (i.e. arc) joining each consecutive crossing pair, to define a number of *faces* (i.e. regions bounded by arcs). Kauffman [5] describes a beautiful algorithm for achieving this.

The problem then becomes which of these faces to choose as the drawing's perimeter; this is the single most important factor in defining the layout and character of the resulting drawing. The following rules suggest which face to choose as perimeter in order to produce a "nice" knot drawing.

Rule 1: Most Perimeter Crossings Choose as perimeter a face with the most crossings. This rule is most important. The perimeter bounds a finite area for the interior of the knot, so any crossing not on the perimeter must add to the knot's internal clutter. For example, the preferred projection of the 3_1 knot (Figure 1, left) has all three crossings on its perimeter leaving an almost empty interior, while the less preferred projection (Figure 1, right) only has two crossings on the perimeter and a more complex interior structure.

More crossings on a knot's perimeter also allows shorter exterior arcs if crossings are to meet at the preferred 90°. Figure 2 shows how having fewer exterior crossings results in excessively long exterior arcs if 90° perpendicularity is enforced (this condition is relaxed in the more elegant drawings of Figure 1).

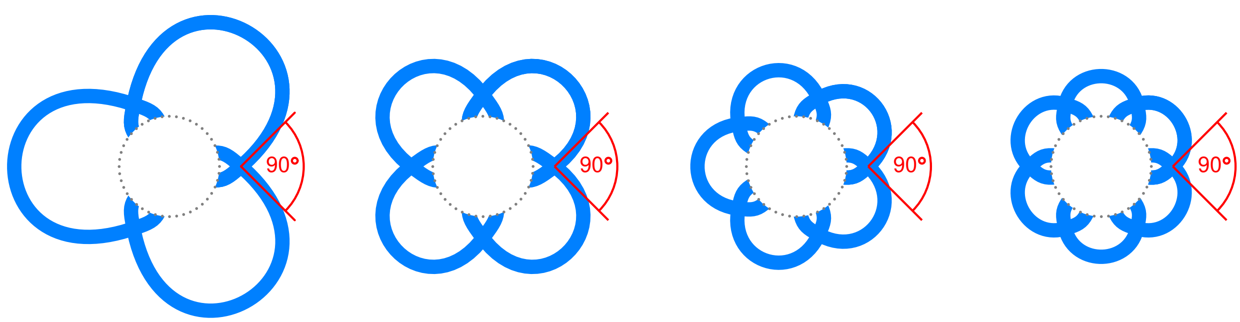

Figure 2: *The effect of enforcing 90° crossings over increasing perimeter crossing counts.*

Rule 2: Most Internal Symmetry Given a perimeter with the maximal number of crossings (Rule 1), then symmetry and repetition can make the internal structure of the resulting knot more comprehensible to the viewer. This can be as simple as bilateral symmetry (ignoring crossing orientation) or the more complex symmetry measures suggested by Generao and Séquin [3]. Note that internal symmetry is not a factor of the perimeter itself and its measurement typically requires at least the construction of the arc diagram. The examples in the following section use simple bilateral symmetry estimates in the range [0..1].

Figure 3: *A 2-face, a 3-face and a 4-face along the perimeter.*

Rule 3: Most 2-Faces and Fewest 3-Faces Given a perimeter with the maximal number of crossings (Rule 1) and good internal symmetry (Rule 2), then 2-faces along the perimeter are to be encouraged while 3-faces along the perimeter are to be avoided. Figure 3 shows a 2-face, a 3-face and a 4-face along the perimeter.

A perimeter 2-face effectively consumes two crossings at negligible cost in terms of comprehensibility. The horizon line of the cord passing through the two crossings could be seen as a perceptual boundary that delineates the outer arc, due to the Gestalt principle of *good continuation* [4]. This allows the viewer to mentally place that arc in the (unbounded) outer region, further simplifying the perceived internal structure.

Unfortunately, the same cannot be said for perimeter 3-faces (Figure 3, centre), which visibly add complexity to the local structure without offering much benefit in terms of symmetry. Perimeter 4-faces (right) are more neutral structures, as the added local complexity is often offset by the local symmetry formed between the 4-face's inner arc and its outer (i.e. perimeter) arc whose curvature it is bounded to follow.

Examples

Figure 4 shows two projections of knot *8nh_2* from the Regina[2] knot database [1]. Both projections have the same number of perimeter crossings (5) and similar symmetry estimates (0.5) but the left projection benefits from having one more 2-face and one less 3-face on the perimeter, making its internal structure more visually comprehensible. The left projection is easier to draw from memory and is arguably the nicer knot.

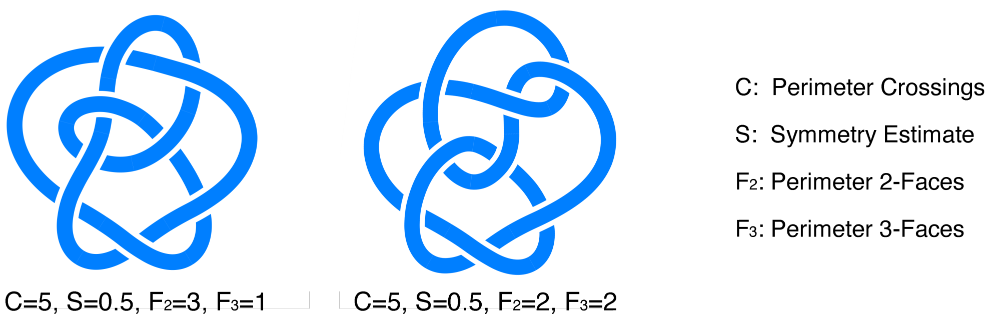

Figure 4: *Projections of 8nh_2 with equal symmetry but more (left) and fewer (right) perimeter 2-faces.*

Similarly, Figure 5 shows three projections of knot *16ah_137806* with the same number of perimeter crossings (5). The left and centre projections are highly symmetrical while the right projection has lower symmetry and is clearly less comprehensible. However, the left projection has two more 2-faces on its perimeter than the centre projection and arguably a simpler internal structure.

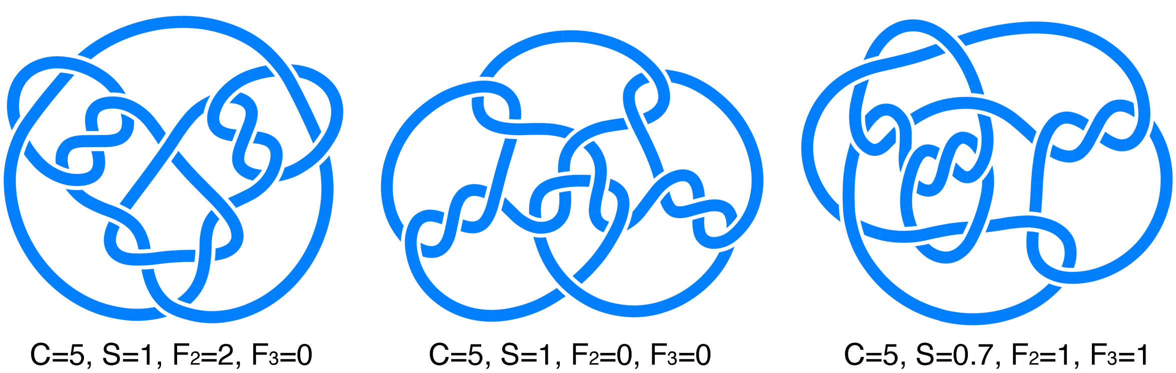

Figure 5: *Projections of 16ah_137806 with high symmetry (left and centre) and lower symmetry (right).*

[2]Regina labels knots by crossing count and whether they are 'a'lternating, 'n'on-alternating, 't'oroidal, 's'atellite or 'h'yperbolic.

Experiments

These ideas were implemented in a Java program and tested on the complete Regina dataset of knots up to size 16. Each knot's bilateral symmetry was estimated by following minimal paths between opposite points on its perimeter and taking the fraction of "symmetrical" steps (through faces with the same local geometry on both sides) divided by the total number of steps. Table 1 shows the results (with duplicates removed).

Table 1: *Number of resolutions per rule for all Regina knots to size 16.*

Size	Knots	1. Crossings	2. Symmetry	3. 2-Faces	4. 3-Faces	Total	Percent	Ties
3	1	1	0	0	0	1	100.00%	1
4	1	1	0	0	0	1	100.00%	4
5	2	2	0	0	0	2	100.00%	2
6	3	3	0	0	0	3	100.00%	2
7	7	7	0	0	0	7	100.00%	2
8	21	15	6	0	0	21	100.00%	4
9	49	40	6	1	2	49	100.00%	3
10	165	113	30	11	8	162	98.18%	4
11	552	348	128	39	17	532	96.38%	4
12	2,176	1,406	474	139	75	2,094	96.23%	6
13	9,988	6,203	2,484	633	372	9,692	97.04%	4
14	46,972	29,142	11,628	3,078	1,611	45,459	96.78%	4
15	253,293	157,372	65,261	15,757	8,437	246,827	97.45%	6
16	1,388,705	864,489	362,749	84,841	42,603	1,354,682	97.55%	8

The second column shows the total number of knots of each size, and the following four columns show the total number of cases resolved to a single preferred perimeter face by applying each rule. The "Total" column shows the total number of knots resolved to a single preferred perimeter face. The high resolution rates over all sizes demonstrate that the method distinguishes a preferred perimeter for the vast majority of cases. All cases checked visually showed good perimeter choices that were at least as "nice" as the alternatives.

Conclusion

The ideas suggested here are not intended for classifying or organising knots, but rather to help standardise the way they are drawn, so that similar Gauss codes produce similar drawings. The approach is straightforward and fast, rarely taking more than 1 millisecond to evaluate any knot in the Regina database up to size 16 on a single thread of a standard laptop machine (including arc diagram construction and symmetry testing).

Future work might involve user testing to validate assumptions about the comprehensibility of generated knots. Additional criteria for comprehensibility might also be interesting to explore, e.g. geometric properties of the curve such as number of inflections of curvature, degree of parallel cord agreement, etc.

References

[1] B. A. Burton, R. Budney, W. Pettersson, *et al.* "Regina: Software for Low-Dimensional Topology." *GitHub*, Version 7.0, 1999–2021. Online: http://regina-normal.github.io/

[2] G. Costagliola, M. De Rosa, A. Fish, *et al.* "KnotSketch: A Tool for Knot Diagram Sketching, Encoding and Re-Generation." *Journal of Visual Languages and Sentient Systems*, 2, 2016, pp. 16–25.

[3] P. A. Q. Generao and C. H. Séquin. "Maximizing the Symmetry of Knots." *Bridges 2020 Conference Proceedings*, 2020, pp. 145–152.

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[5] L. H. Kauffman. "Virtual Knot Theory." *European Journal of Combinatorics*, 20, 1999, pp. 663–690.

[6] P. Rideout. "Generating SVG for the Prime Knots." *The Little Grasshopper*, 2020. Online: https://prideout.net/blog/svg_knots/