Composite Number Polyrhythms: Animating and Sonifying the Divisor Plot

Jeffrey Ventrella
jeffrey@ventrella.com

Abstract

The divisors of composite numbers generate an endless complex pattern when plotted perpendicular to the number line as a function of size. This pattern—the *divisor plot*—is an infinite set of 2D points with integer values that can be described as a visualization of the Sieve of Eratosthenes. It has a lot of curious patterns, but if we include the element of time (with animation or *sonification*), even more patterns are revealed, as well as polyrhythms. When interpreted as a *musical score*, the divisors present a large palette of parameters for generating *minimalist math music*. This paper describes the discoveries, experiments, and design methods I have used to transform this mathematical canvas into visual and sonic expressions.

Introduction

A vast ocean of complexity lies hidden inside the integer number line. We usually don't see it because we are looking at only one attribute of numbers: *size*. I hereby proclaim that the size of an integer is its most boring attribute! The number 181 is 1 more than 180, but it is a prime number while 180 has 17 proper divisors. The number 180 is the 14th *highly-composite* number (having more divisors than any smaller number). Its divisors include a contiguous chunk from 1 to 6; they form a unique fingerprint, related to the fundamental theorem of arithmetic.

The number 180 is flanked by twin primes (179 and 181). It can be appreciated in the context of the divisors of its neighbors, extending well beyond its twin prime companions. This neighborhood can be visualized as a structure in 2 dimensions, where the vertical dimension is used to plot the divisors of each integer, as a function of the divisor's size. This is illustrated in Figure 1, showing a region of the divisor plot [14] centered at the number 180. A window of divisors < 17 is shown for the integers from 162 to 197. Primes are highlighted with vertical bands. The figure shows how the string of divisors of 180 is "projected" as rays on either side, revealing a symmetrical pattern that is not apparent in the one-dimensional representation of the number line.

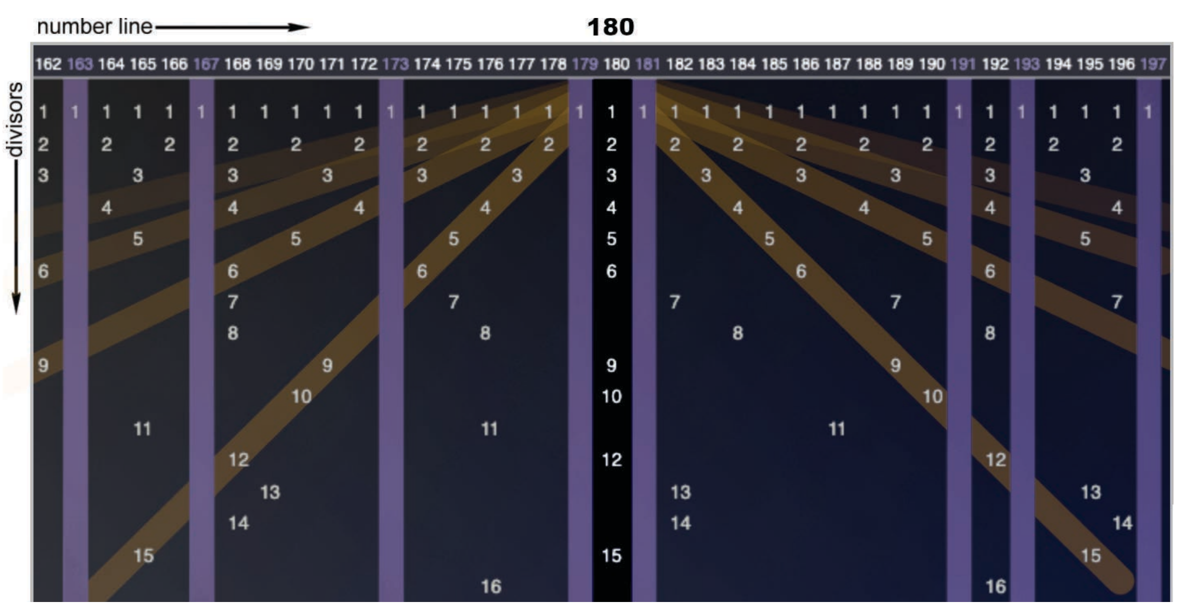
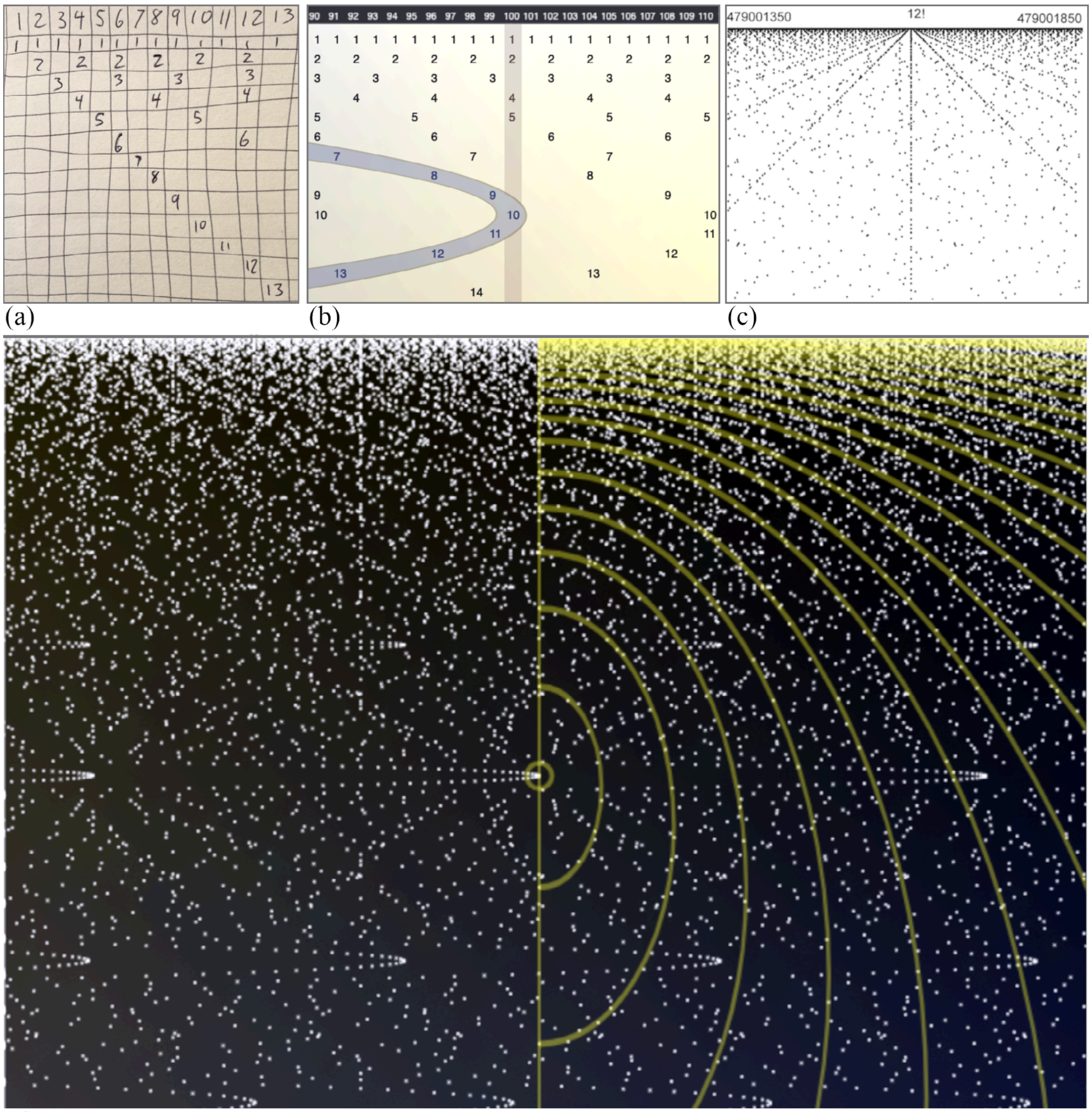


Figure 1: *The divisors < 17 of 180 and its immediate neighbors. Primes are shown as vertical bands.*

As the figure suggests, the number 180 has a lot more going on than the number 181, which makes it *bigger* in a sense…kind of like a complex chord in a musical composition.

I discovered this pattern at an early age while exploring polyrhythms with hand-drawn diagrams. I plotted several rows of numerals by hand, spacing-out the numerals in each row according to its vertical position (Figure 2a). Upon seeing some unexpected patterns, I felt as if I were an ancient astronomer, looking for meaning by connecting the dots. Some examples of patterns later found in the divisor plot include a parabola of ordered divisors centered at the square root of 100 (Figure 2b); a window on the neighborhood of 12! (twelve factorial = 479,001,600) (Figure 2c); and a parabola centered at the square root of 1 million, showing the divisor pairs of 1 million, connected with yellow arcs (Figure 2d).



(a) (b) (c) (d)

Figure 2: *Four views of the Divisor Plot: (a) 1-13; (b) a parabola at n=100; (c) 12 factorial; (d) a parabola at the square root of one million.*

The divisor plot can be defined as the set of points in the (n, d) plane, where n and d are positive integers forming a square lattice, and where $n \bmod d = 0$. (n refers to the integers and d refers to their divisors). I prefer to flip the d axis so that larger divisors appear lower in the plot.

I presented the divisor plot at the 7th International Conference on Complex Systems in Boston. Soon after, I created the first version of an online exploration tool (divisorplot.com), and have since corresponded with fellow number pattern fans from around the world who have made similar discoveries, usually in the process of searching for the holy grail. Many of them have made elaborate, colorful hand-drawn graphs and diagrams—each one revealing a unique and highly personal relationship with the primes. The on-ramp of discovery does not require much mathematical knowledge: a thoughtful doodle might be the initial seed. For pattern-lovers, this is a doorway to number theory: a path of discovery that never ends.

Music of the Composites

Instead of thinking of primes as *building blocks*, think of them as *empty gaps* in a forest of overlapping composite patterns. And instead of searching for the "music of the primes", as Marcus du Sautoy suggests [6], consider primes as silent moments in a never-ending symphony of divisors, marching along the number line. According to Gregory Chatin, "A concept is only as good as the theorems that it leads to! ... instead of primes, perhaps we should be concerned with the opposite, with the maximally divisible numbers!" [3].

Now, you may ask, what is the "music of the composites?" Think of the divisor plot as a musical score. If we choose to "play" the divisors of composite numbers instead of the primes, we are rewarded with a large palette of sound parameters—and they are all relatable to number theory. In this paper I will show that we have many parameters, knobs, and levers to tweak, constituting a large conceptual space for creative expression, both in the audible and visible realms.

Background

The parabola patterns shown above have been analyzed by David Cox [4]. Other visual representations of integer divisibility are shown in Figure 3. These include the Ulam spiral [7] (typically showing primes but in this illustration showing the number of divisors) (Figure 3a); the Sacks prime spiral [9] (Figure 3b); a pattern from Shekatkar, et. al. [11] (Figure 3c); an illustration from Wolfram [17] (a smaller, earlier version of the pattern I discovered) (Figure 3d); factorization diagrams (numbers expressed as dot patterns) by Brent Yorge [18] (Figure 3e); and the "Divisor Stack Spiral", a 3D design by Dan Bach [1] (Figure 3f).

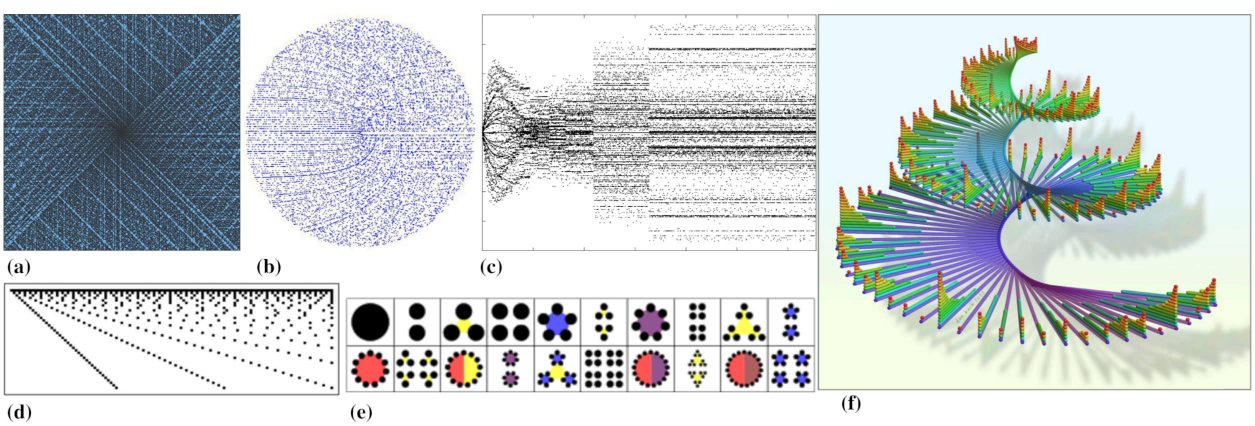


Figure 3: *Patterns of divisibility: (a) Ulam spiral; (b) Sacks spiral; (c) a pattern by Shekatkar, et. al; (d) divisor pattern from Wolfram; (e) factorization diagrams, and (f) a design by Dan Bach.*

Many more visual number patterns have been created by Stange [12], Paris [8], and Croft [5], to name only a few. Regarding *temporal* expressions of integer divisibility along the number line, these are more prevalent in music than in animated visual art, possibly because these kinds of patterns are better-appreciated by ear than by eye. Polyrhythms are among the oldest human manifestation of integer divisibility, by way of overlapping sonic events with different integer periods. These are famously demonstrated in many forms of traditional African music. Indian classical music is known for complex polyrhythms, often including the prime number period 5. Wannamaker describes complex layered polyrhythms in a composition by Conlon Nancarrow for player piano [15], and Toussaint shows how traditional rhythms can be generated via the Euclidean algorithm [13].

How much expressivity can be generated or "extracted" by scrolling through the number line? If the raw data being expressed are divisors (not the numbers themselves), then we have a large palette to work with.

Animation

The longstanding quest to understand the distribution of primes is invigorated by new research that emphasizes dynamical systems, chaos, and concepts from physics. Here, the element of time is invited into the imagination, bringing up deep questions about the "origins" of the number patterns we are trying to understand [16]. Animation can be used to leverage intuition, as demonstrated by provocative animated visualizations of the Riemann zeta function, which generates primes in the complex plane [18].

A picture is worth a thousand words, and an animated graph is worth a thousand numbers. I came to the realization that the divisor plot could express a lot more than what can be seen in any single static image…somewhat by accident. I had decided to add a scrolling feature to the divisor plot web site so viewers could watch new patterns drift by as the viewpoint slowly scrolled rightward along the number line. On a whim, I decided to set the scrolling rate to a (very) high number. I was immediately struck by amazing patterns that issued forth, shifting and undulating in many directions.

What usually constitutes a nuisance in animation craft (strobing and aliasing artifacts) became a tool of mathematical exploration. It was obvious in retrospect: the frame-rate at which the animation was running interacted with the modular nature of the divisor patterns as they zipped by—often locking into a kind of illusionistic stability. What was the generator of that stability? It certainly wasn't the primes.

Exponential Animation Jumps

Scrolling along at a steady rate reveals interesting dynamical patterns, but scrolling at an *exponential* rate reveals even more wonders. The parabolas shown in Figure 2 are in fact members of an infinity of parabola families that are distributed exponentially. They are strewn throughout the divisor plot, growing in number and neighborhood-complexity along the number line. Parabola vertices can be found at (n, d) where $n = d^2a$. Figure 4 shows a few frames from an animation where d increments by 1 for each animation frame, and where $a = 2$, allowing the view to fix on the scrolling vertices of a parabola series. In this animation, the divisor plot is rotated by 90 degrees, creating a vertical orientation. The view scrolls "upward" along the number line, as if a fleet of rockets were blasting up through a starry sky.

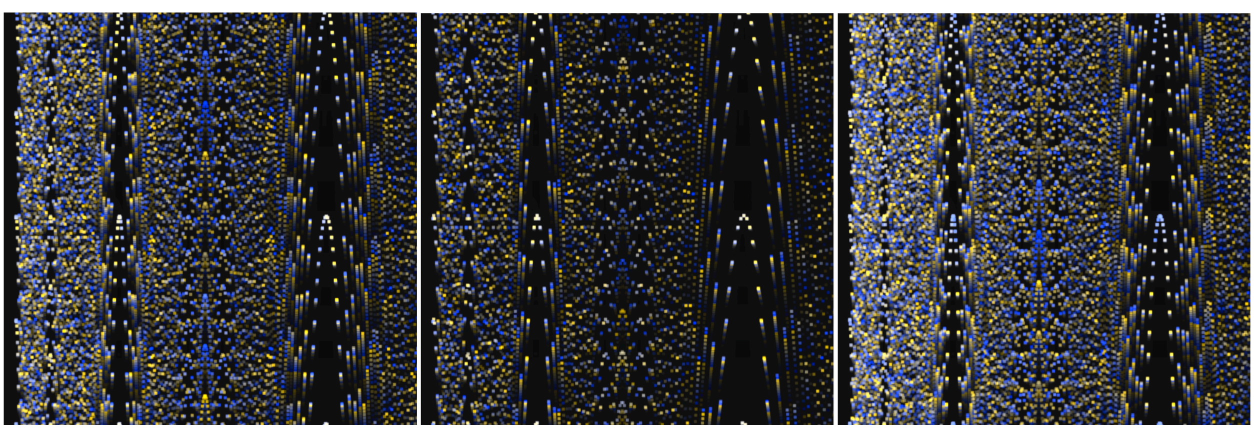


Figure 4: *Three stills from a scrolling animation. The divisor plot is rotated by 90 degrees.*

Sonification

Imagine an ensemble of six drummers playing six congas. The first drummer plays a regular, steady beat, striking a conga at a rate of about four beats per second. The second drummer strikes a conga every two beats. The third drummer strikes every three beats, and so on. If all six drummers started playing at the exact same time, then on that first beat, all drums would be playing simultaneously, followed by something that bears a resemblance to chaos. So here's a question: will all six drums ever play simultaneously again? Figure 3 shows each drum beat as a black square, with time moving to the right, and with drummers ordered from top to bottom. It reveals that all drums will be played simultaneously every 60 beats. Notice the palindromic symmetry with beat 30 on the axis of reflection. Larger variations of this symmetrical pattern are associated with the factorials, in which all the drums are guaranteed to play simultaneously every d! beats, where d is the number of drummers.

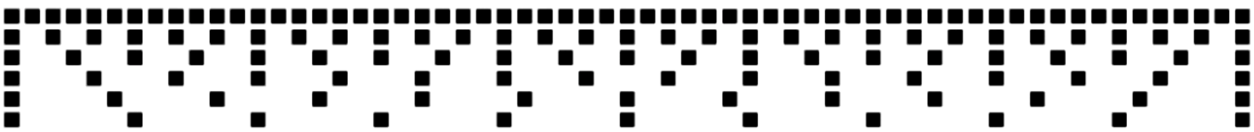


Figure 5: *Polyrhythm of six drummers (ordered from top to bottom) each playing at a different rate.*

If we were to omit the first drummer (Figure 3, top row), then we would notice silent spaces: beats where no drums are being played. It's easy to see that if this diagram were extended in time (rightward) and in terms of the number of drummers (downward), it becomes a visualization of the sieve of Eratosthenes—one of the oldest algorithms used for finding prime numbers. It also happens to be an exact replica of the divisor plot pattern.

Here's a puzzle that lies at the intersection of music and math: Imagine an arbitrarily-sized set of overlapping sound events played over time at different integer periods. What kinds of polyrhythms and syncopation are possible? I will not attempt a precise answer, but I will say…*many!* To explore this question, I made an experimental version of the divisor plot that generates sounds—a kind of polyrhythmic drum machine (Figure 6), available online at divisorplot.com/polyrhythms.

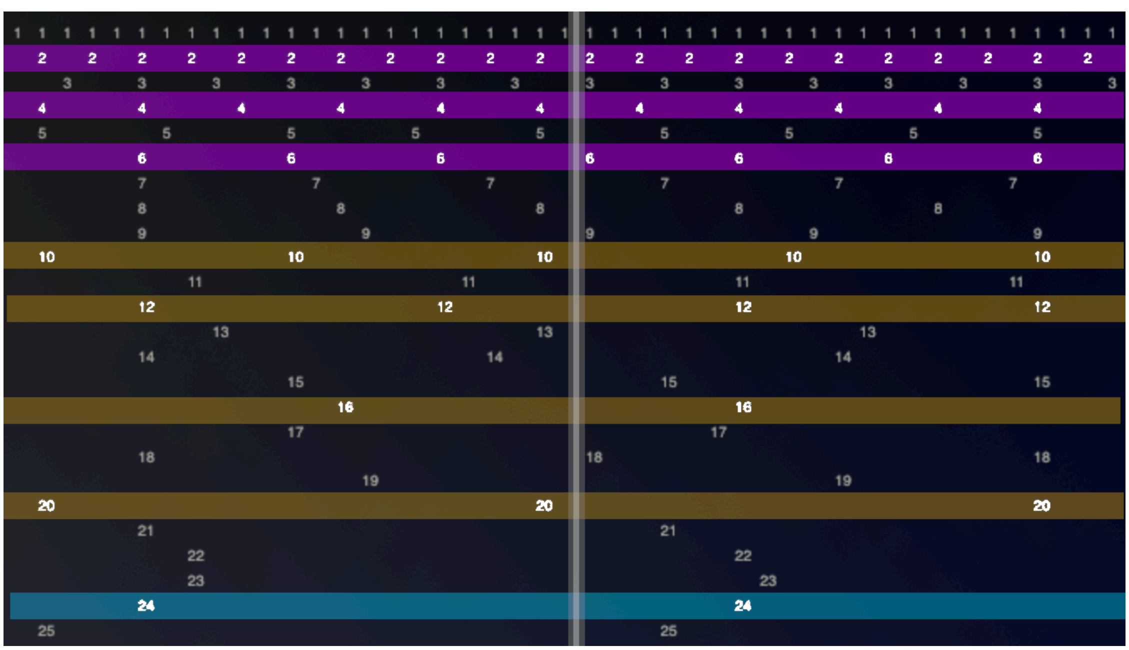


Figure 6: *A polyrhythmic drum machine based on the divisor plot.*

The drum machine uses up to 12 “voices”. Each voice has an associated pitch, volume, duration, instrument, and *divisor*. Think of these voices as sound-enablers that can be positioned on any one of the divisor rows. The center line shown in the illustration serves as a metaphorical *recorder tape head* that plays the sound of a divisor as it scrolls from right to left and comes in contact with the center line. Whenever a divisor with a voice is played, a horizontal band appears in a color that indicates which instrument is playing. For illustration, Figure 6 shows all the colored bands from a rhythm I designed.

This rhythm (see “12 Groove” at divisorplot.com/polyrhythms) is the result of the last of five basic experiments. For most of these experiments, percussive sounds (no clear pitch) were used. Some of these sounds are very short-lived “sound points”, intended to serve as the sonic equivalent of a dot in a graph.

1. Simple polyrhythms

As indicated in Figure 5, it is easy to generate simple familiar polyrhythms, such as 2 against 3, 3 against 4, or (2, 4, 5). For this experiment I simplified the sound parameters to act as sound-points; I wanted to avoid musical emotion and just “hear the math”.

2. Powers of Two

Have you ever wondered how to make a polyrhythm with as little syncopation as possible? A power-of-two polyrhythm (1, 2, 4, 8, 16, 32) seems to do the trick. For this experiment, I used instrument sounds with pitch intervals (e.g., whole-tones, 4ths…) that are inversely-proportional to divisor size so that more frequent sound events are higher in pitch than infrequent ones. It came across as a kind of march: I imagined a parade of many-sized creatures, from slow-plodding elephants to fast-stepping mice.

3. Factorial Waypoints

A factorial *n*! is the product of the first *n* numbers. In the divisor plot, factorials are visually prominent, as apparent in Figure 2c. The solid vertical string of divisors in a factorial is associated with the simultaneous playing of every divisor (a “factorial waypoint”). Even miniature factorial waypoints can be experienced, such as 4! When configured with a well-chosen set of percussive timbres, 4! creates an interesting 24-beat cycle.

4. Primorial Chaos

Primorials are like factorials except that instead of being the product of the first *n* numbers, they are the product of the first *n primes*.What do you think you would hear if the only divisors played were the first 5 primes (2, 3, 5, 7, 11)? The product of these primes (the primorial denoted as $P_5\#$) is 2,310. These 5 divisors will all play simultaneously every 2310 beats.

So, what does it sound like? It sounds horrible. On the other hand, if you like sounds that are devoid of any recognizable pattern, then perhaps it might satisfy your intellect. Playing the prime divisors up to 41 at a high tempo (with each divisor generating an identical sound-point) creates the pitter-patter of rain on the roof…or perhaps a hail storm. Similar to the power-of-two experiment, I made *volume* inversely-proportional to divisor size. Figure 7a shows the divisor plot with prime divisors up to 41 highlighted. Figure 7b shows a vertically-compressed view where only the primes are shown. Note: $P_{41}\#$ will repeat every 304,250,263,527,210 beats.

(a) (b)

Figure 7: *Primorial Chaos:(a) prime divisors highlighted; (b) compressed view with just prime divisors.*

5. Calculated Syncopation

The polyrhythm shown in Figure 6 refers to a simple rhythm I designed. Besides the wonderful composite structure already inherent in a number like 24 (factorial 4!), I added some off-beat periods like 10, 20, and 30, for a bit of sparkle. These periods have 5 as a prime factor, so they interact with the prime factors of 24 (2 and 3) to make some surprising syncopation. The sound parameters of these off-beat periods were configured to function as subtle, high-pitched percussive accents.

Minimalist Math Music

The music of composer Steve Reich makes use of overlapping and phasing patterns in time. These patterns are repetitive and typically slowly-transforming. When compared to the melodies of Irving Berlin or the dynamic range of Stravinsky, Reich's music may seem mindlessly repetitive. But this impression fades as the music induces a sort of trance that puts the listener into a different relationship with time, a key aspect of minimalist music; it exists in *vertical time*. Styles such as Trance and Ambient do not have long, Beethoven-like arcs of thematic development, where the ending serves as a homecoming. Consider the traditional mbira music of Africa, with cross-rhythms and rich timbres…and no clear beginning or ending.

An example of endless algorithmic music is Pi-Songs, by Canton Becker [2]. The software reads a continual stream of the digits of Pi to algorithmically generate ambient music from a large palette of sounds. The listener is invited to "fast-forward", or "rewind" to any digit of Pi to revisit favorite spots. Since the digits of Pi are essentially random (though predictable and repeatable), it permits the artist to craft an interpretation with control over a mathematical pattern that has no inherent structure. The divisor plot, in contrast, constitutes a maximally-structured musical canvas. The degrees of artistic freedom are constrained in a way that depends on which divisors are expressed and how they are interpreted sonically.

Harmonic structure

Now think of the string of divisors hanging below each number as the notes of a musical chord. Some "chords" have only two notes (the primes). Others have lots of notes. Does there exist a mapping from divisors to musical pitches that generates every possible chord? A more artistically-productive question is: might there be a mapping that generates meaningful music along the number line?

"Meaningful" is the operative term; it would be easy to generate utter cacophony by simply assigning each divisor to a unique pitch, where some arbitrary divisor is assigned to a specific pitch (like A440) as a baseline. Starting from 1 and scrolling at a rate of 4 numbers per second, it may sound interesting for a few seconds, but then chaos will set in…and it will not abate. However, a process of filtering out select divisors, and applying a modulus on the pitch to constrain its range may result in something more interesting. But is it melodic?

Not yet. At this point, we can imagine applying any number of arbitrary constraints and clever mappings to make it more musical, including voices synchronizing divisors or instruments, turning on or off, or changing divisors. For me, the primary goal of such an exercise is to find out how much music can be extracted from something as precise, predetermined, inherently-structured, and mathematically fundamental as the factors of the natural numbers. If something comes out that is sonically nourishing while also expressing something beautiful about number theory, then maybe we can call it art.

Conclusions

One can analyze, measure, and precisely describe a perceived object hidden in the number line, such as a parabola. One can also *synthesize* novel expressions of that precise mathematical object by mapping it to the senses, using position, color, movement, and sound—in a thousand different ways. As an algorithmic artist, I am preoccupied with how to map these Platonic mathematical forms into something that fills the senses and feeds the organs of intuition; to go deeper than merely feeding a pattern fetish. Even if we never crack the code of prime numbers, the process of searching can lead to new insights. Composite numbers provide something that the primes do not: beautiful structure at every scale. If the primes are the leftover result from taking away all the patterns, then we should get to know the patterns.

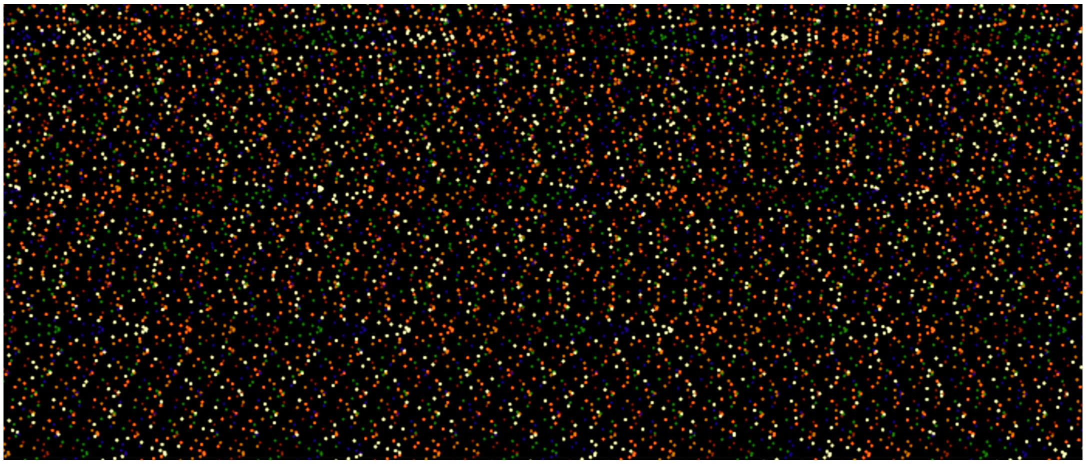


Figure 8: *A view of parabola streams with a chosen color palette.*

Acknowledgements

I want to thank the many obsessed number pattern seekers with whom I've shared ideas and diagrams over the years.

References

[1] D. Bach. Divisor Stack Spiral, 3D Design, URL as of 2/1/2022: https://1-dan-bach.pixels.com/featured/divisor-stack-spiral-dan-bach.html
[2] C. Becker. Pi Songs. URL as of 2/1/2022: https://pisongs.com/
[3] G. Chatin. *MetaMath! The Quest for Omega*. Vintage, 2006.
[4] D. Cox. "Visualizing the Sieve or Eratosthenes". Notes of the AMS. Volume 55, number 5. 2008.
[5] G. Croft. The Prime Spiral Sieve. URL as of 2/1/2022: https://www.primesdemystified.com/
[6] M. du Sautoy. *The Music of the Primes*. Harper, 2012
[7] M. Gardner. (March 1964), "Mathematical Games: The Remarkable Lore of the Prime Number", Scientific American, 210: 120–128, doi:10.1038/scientificamerican0364-120
[8] C. Paris. Primal Chaos (Visualizations). URL as of 2/1/2022: http://www.sievesofchaos.com/
[9] R. Sacks. The Number Spiral. URL as of 2/1/2022: https://numberspiral.com/
[10] G. Sanderson. Visualizing the Riemann zeta function and analytic continuation. URL as of 2/1/2022: https://www.3blue1brown.com/lessons/zeta
[11] D.M. Shekatkar, et.al. "Divisibility patterns of natural numbers on a complex network". arXiv: 1505.01694 [cs.SI]
[12] K. Stange. NumberScope. URL as of 2/1/2022: https://math.katestange.net/numberscope/
[13] G. T. Toussaint. "The Euclidean algorithm generates traditional musical rhythms". *Proceedings of Bridges: Mathematical Connections in Art, Music, and Science*. 2005, pp. 47–56.
[14] J. Ventrella. *Divisor Drips and Square Root Waves*. Eyebrain Books. 2010. URL as of 2/1/2022: https://www.divisorplot.com
[15] R. Wannamaker. "Rhythmicon Relationships, Farey Sequences, and James Tenney's Spectral CANON for CONLON Nancarrow". Music Theory Spectrum, Vol. 34, Issue 2, pp. 48–70, ISSN 0195-6167, electronic ISSN 1533-8339. © 2012 by The Society for Music Theory.
[16] M. Watkins. Number Theory and Physics: URL as of 2/1/2022: https://empslocal.ex.ac.uk/people/staff/mrwatkin/
[17] S. Wolfram. *A New Kind of Science*. Wolfram Media, 2002.
[18] B. Yorge. Factorization Diagrams, URL as of 2/1/2022: https://www.exploratorium.edu/blogs/tangents/composite-patterns