Algorithmic Approach to Triangular-twist Tessellations: Adding Spacings as a Creative Method

Ekaterina Pavlovic (Lukasheva)

art@kusudama.me

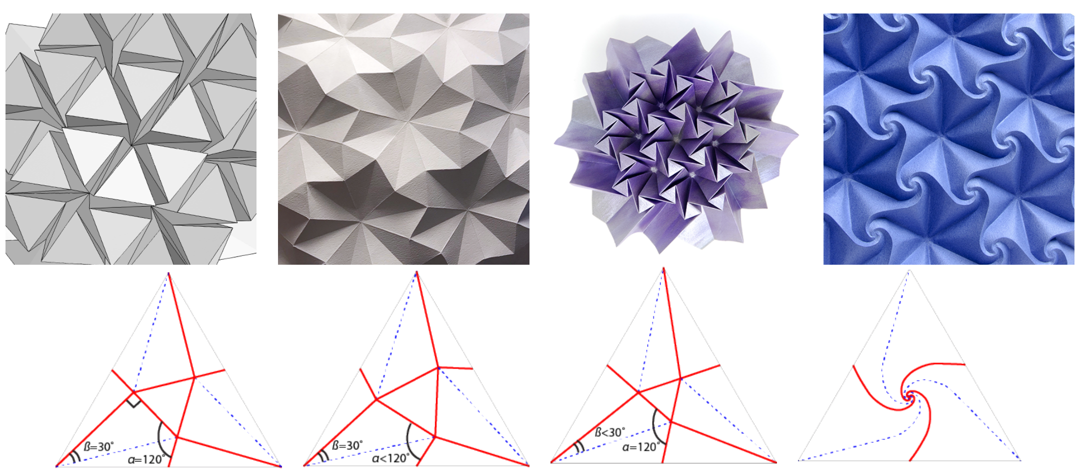
Abstract

This paper explores design possibilities of adding spacings to a whole class of non-flat triangular twist origami tessellations. Convenient molecule (building block) definition is introduced which allows adding spacings easily. Two main methods of adding spacing are discussed, the taxonomy is proposed for the first type. Combinations of two types are discussed along with some examples and corner cases.

Introduction

Origami tessellations are repeating structures folded from a single sheet of paper without the use of cuts. This paper considers a subclass of non-flat origami tessellations and explores adding spacing as a design method for finding new interesting patterns. I introduce a convenient way of dissecting an initial design and attempt to describe all possible variants of added spaces to these tessellations.

Spatial, non-flat polygonal twist tessellations were explored by David Huffman [1] and Ron Resch [5] and later covered by Tomohiro Tachi as so called “Generalized Resch“ tessellations [6]. Triangular twist tessellations is a subclass of this group. One of the best known examples is a tessellation by Ron Resch (see Figure 1a), which can be further generalized to the whole class referred as “Generalized Resch“ tessellations [6]. One of the examples of such tessellation is shown in Figure 1b. It is easy to show that spiral tessellations like one in Figure 1d belong to the same class of triangular twists and have the same physical and geometrical characteristics as their straight-fold peers.



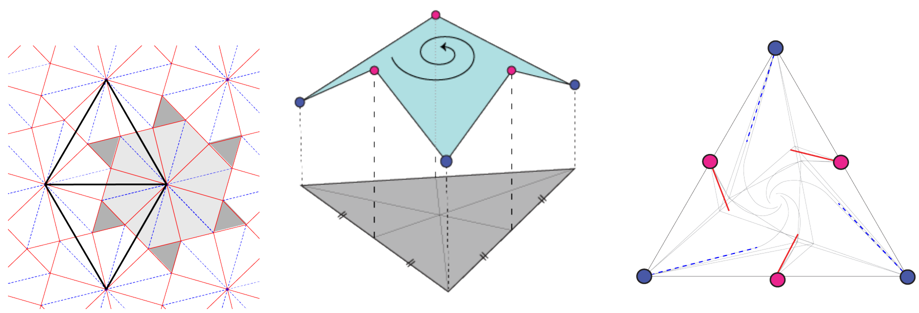
(a) **(b)** **(c)** **(d)**

Figure 1: *Triangular-twist tessellations and their molecules: (a) Ron Resch (simulation [2]), (b) Generalized Resch, (c) Sharp rose opus T-165 by author, (d) Mermaid-B opus T-37 by author.*

Molecule Definition

A *molecule* of a tessellation is an area, usually a polygon, that, if repeated, can tile the whole plane creating that tessellation. While there are different ways of defining molecules for any given tessellation, I will show that defining them as triangles makes the design process easy and straightforward.

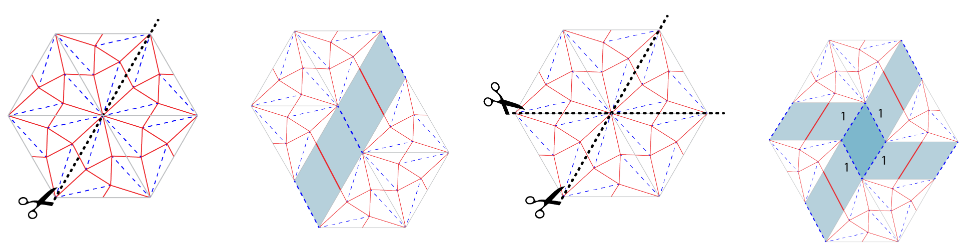
The usual way of working with twist tessellations involves separating the flat polygons that show on the surface of a tessellation from the polygonal or star-shaped elements that are attached to them [6], see Figure 2a. Let us define a molecule in a different way: let us consider triangles with vertices in the lowest points of the folded tessellation, i.e. in the middles of a hexagons (or hexagonal stars). This way we get equilateral triangles such that the top triangles of the folded tessellation are in the middles of them. The mountain folds will cross the sides of these triangles in the midpoints of each side. This way the vertices of the molecules become the lowest points of a folded tessellation and the highest points are located in the midpoints. Between these points the border of a molecule consists of straight segments. This fact allows for seamlessly connecting the molecules or adding spacing between them as discussed later.



(a) **(b)** **(c)**

Figure 2: *(a) Old (shaded grey) and new (bold lines) molecule definitions for Resch tessellation: solid and dotted lines represent mountain and valley folds respectively, (b) schematic spatial view of a general case molecule, (c) several triangular molecule crease patterns aligned.*

Adding Spacing



(a) **(b)** **(c)** **(d)**

Figure 3: *Adding spacing: (a) a single cut on a molecular plane, (b) crease pattern of a single direction spacing added, (c) two intersecting cuts, (d) the resulting crease pattern with equal spacing added along each cut.*

Let us look at a crease pattern of a tessellation and let us cut it along the molecule borders (see Figure 3a). The side creases of all the molecules are going out at the same angle, so we can shift the parts away in that direction, adding a strip of paper in between molecules and connecting the side creases with straight lines (see Figure 3b). Let us connect the vertices of the molecules with valley folds. These valley folds will be parallel to the straight mountain folds we just created. Since molecule sides and the newly created folds are all straight, the paper between the creases and the molecules will create flat areas, slanted at some angle. It is easy to show that a spacing between each two molecules is a parallelogram with two valley folds at the sides and one mountain fold in the middle.

Let us now cut the tessellation twice in two different directions. Then these custs have an intersection point, see Figure 3c. For the sake of simplicity let us add equal spacing along each cut. Unequal spacing will be discussed later. As we connect the lines and add parallelograms between the molecules, we can see the new flat rhomboid shape appears where the spacings added in different directions intersect, see Figure 3d.

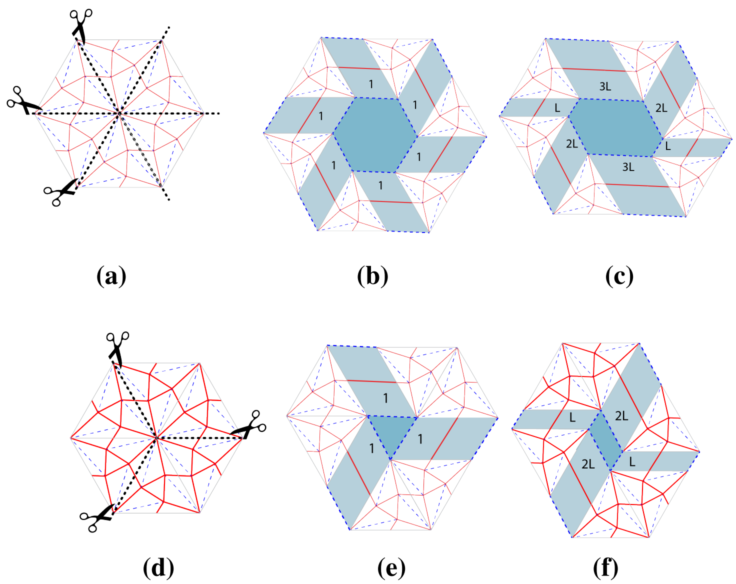


Figure 4: *Three-way cuts and spacing: (a) three intersecting cuts, (b) equal spacing added, (c) unequal spacing added, (d) special case three-way cut, (e) spacing added to a three-way cut, (f) two unequal spacings added in two directions.*

Let us consider three intersecting cuts. See Figure 4a. In case we add the same length spacing, we get a flat hexagon at the intersection area, see Figure 4b. If we consider unequal spacing: with lengths L, $2L$, and $3L$ (where $L > 0$) we will get a hexagon with parallel opposite faces with lengths L, $2L$, and $3L$ accordingly, see Figure 4c. In case of two-cut intersection with different length spacing, we will get a parallelogram instead of a rhombus, see Figure 4f.

Special case: the three-way cut

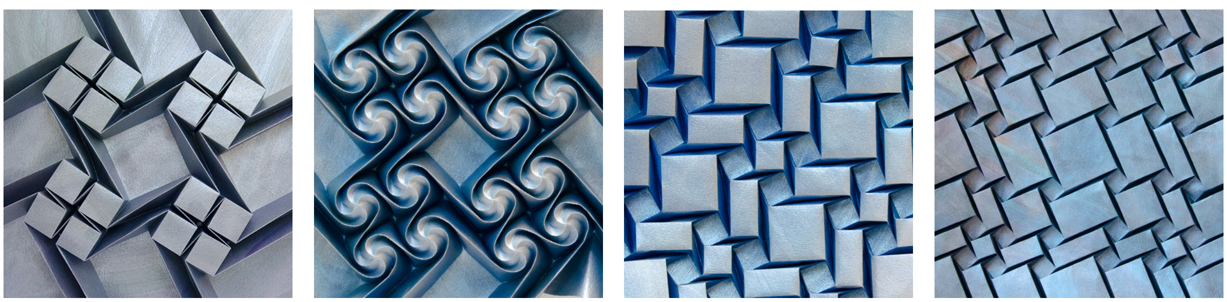
Let us consider the special case: instead of cutting all the way through, let us cut like shown in Figure 4d and then try to add parallelograms to the molecules as we did previously. We can easily see that this is the only case where lengths of spaces should all be equal, otherwise you can't connect the pieces back. The resulting crease pattern with an equilateral triangle in the spacing intersection point is shown in Figure 4e.

Spaced Tessellation Taxonomy

Let us try to systematize and describe tessellations achieved by adding spaces. Pavlovic [4] has previously shown that in case of square molecule twist tessellations the multitude of all the variants of tessellations that can be achieved by adding spaces can be described with two sequences of numbers $(x_1, \ldots x_n)$ and $(y_1, \ldots y_n)$ where $x_i, y_i \geq 0$ are the lengths of spacing in two different directions accordingly and each sequence is endlessly repeated. She has also shown that these sequences are invariant to rotation, meaning that the sequence $(x_i+1, \ldots x_n, x_1, \ldots x_i)$ will describe the same tessellation as the sequence $(x_1, \ldots x_n)$ because it does not matter where to start a sequence, as long as this sequence is repeated an infinite amount of times. Some examples of square twist tessellations are shown in Figures 5a and 5b.

The lengths of spacing can be measured in any units, but it is convenient to align the measurement system to molecule lengths in some way. Each molecule has the polygon length on a crease pattern, i.e. in an unfolded state. This is a very easy length unit, however from the design perspective it may be more convenient to use the length of a folded molecule instead. This way if we use multipliers of the folded triangle length we get molecules nicely aligned in a folded model. The length of a fully folded Resch molecule from Figure 1a will be the length of a top triangle in the molecule. While Generalized Resch molecule from Figure 1b can theoretically be collapsed to the size of its central triangle, that would cause the tessellation to curl and become concave from the front side. Let us instead call the *fully collapsed* state for that tessellation the level of collapse, when the tessellation becomes relatively flat, i.e. not concave or convex. In that position all the top flat triangles will be lying in one plane. From now on I will refer to spacing lengths of 1, 2 etc., where one means the length of one fully collapsed molecule. For each particular molecule that size can be different.

There is one more design perspective to consider. When original Resch molecule is fully collapsed, the tessellation becomes not very interesting aesthetically, as everything except for the top triangles disappears. For this reason I prefer to use Generalized Resch molecules or curved molecules in my designs, as they allow to show the beautiful patterns of the side walls even when fully folded.



(a) **(b)** **(c)** **(d)**

Figure 5: *Spaced square-twist tessellations: (a)* $(2,0)(2,0)$ *spacing added to Huffman/Resch water bomb tessellation, (b)* $(1,0)(1,0)$ *spaced Quadra-D molecules, opus T-219, (c) water bomb molecule with* $(1,2)(1,2)$ *spacing from the back: intersecting spacing rectangles are the top parts in the picture, opus T-208, (d)* $(3,2,1,2)(3,2,1,2)$ *spaced water bomb (back view), opus T-239.*

Straight-cut Spacing Tessellation Taxonomy

Let us try to describe triangular spacing in a similar fashion. For an example let us consider the set of sequences $(x,0)(x,0)(x,0)$, where $x > 0$. In this example 0 denotes the absence of spacing at a certain line. We can place the first two sequences to the molecule plain without any variants, but as we are placing the third sequence, we can see that there are two variants to do so. We can align the spacing of length x so that it comes through the intersection of cuts in first and second direction (see Figure 6d: three cuts intersect at the point marked with a circle). We can also align it the way shown in in Figure 6g so that three cuts never intersect in the same point, i.e. the third direction cuts are shifted from an intersection point by one molecule. Let us assign the sequence $(x,0)(x,0)(x,0)$ to the first variant, signifying by x in the first places that the spacings of length x intersect in same points. And let us assign the sequences $(x, 0)(x, 0)(0, x)$ to the second one, showing that the intersection point of cuts in first and second directions corresponds to the absence of spacing (spacing equal to 0) in the third direction. This means that for triangular case the sequences of spacings aren't completely rotation-invariant like in case of square molecules. We can easily see that if you rotate sequences all at once, they will represent the same tessellation. For example $(0, 2)(0, 2)(0, 2)$ will be equivalent to $(2, 0)(2, 0)(2, 0)$ but not to the $(2, 0)(2, 0)(0, 2)$ as it can be seen on Figures 6f and 6i.

The attentive reader may mention that the shapes between the cuts on molecular plane become molecule cluster groups in a complete tessellation, see Figure 6. This observation helps plan designs in advance.

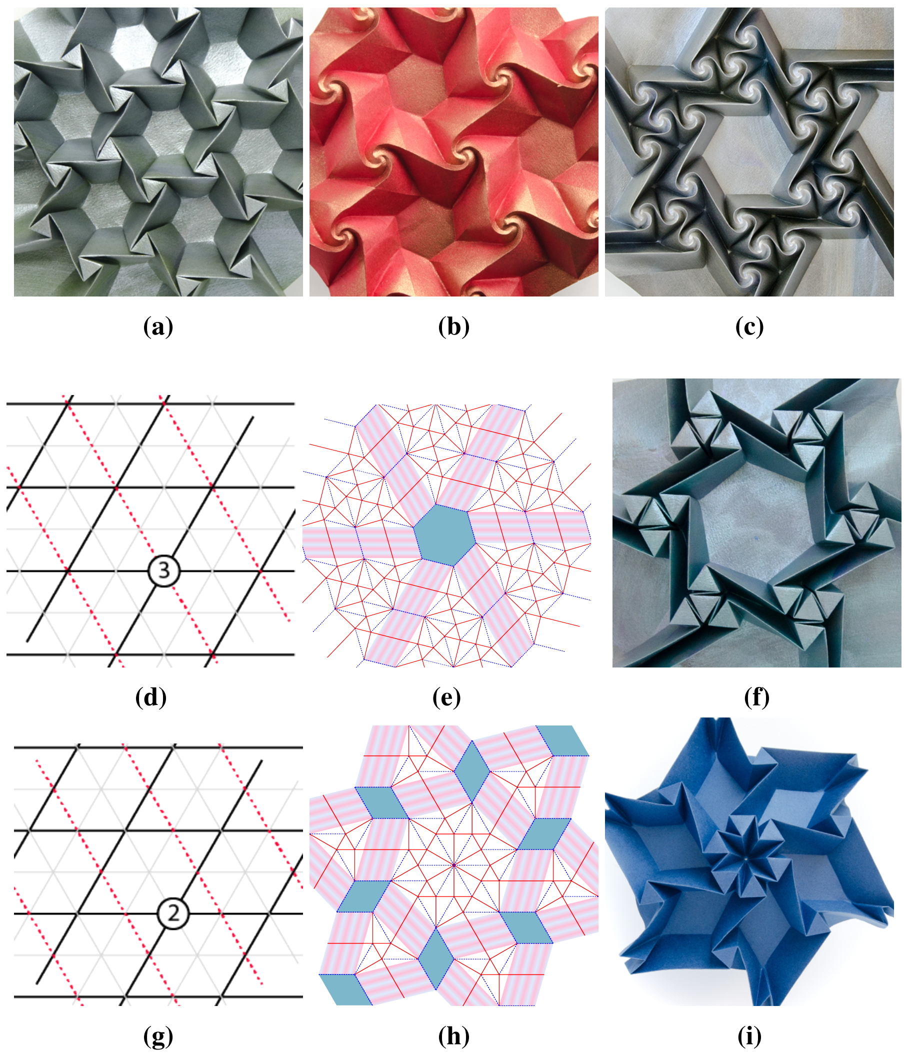
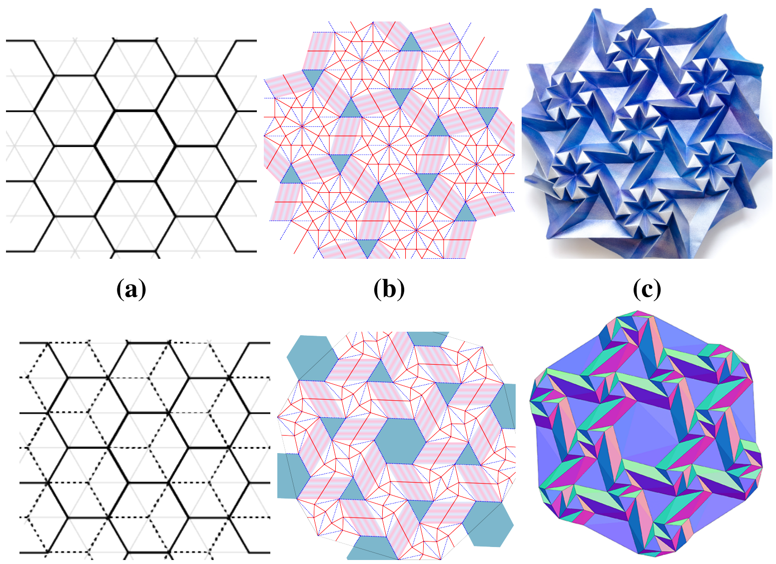


Figure 6: *Spaced triangular twist examples: (a) Sharp rose molecule with equal spacing in all three directions* $(\frac{1}{2})(\frac{1}{2})(\frac{1}{2})$*, (original on Figure 1c), (b) Mermaid* $(\frac{1}{2})(\frac{1}{2})(\frac{1}{2})$ *(original on Figure 1d), (d) Spacing schema for* $(x, 0)(x, 0)(x, 0)$*: the third direction cuts are added in dashed lines, (e,f) resulting crease pattern and the tessellation made for* $x = 2$ *from Resch molecule, opus T-228, (d) Mermaid* $(1, 0)(1, 0)(1, 0)$*, opus T-217 (original on Figure 1d), (g) spacing schema for* $(x, 0)(x, 0)(0, x)$*, (h,i) the resulting crease pattern from Resch molecule for* $x = 2$*, opus T-233.*

Hexagonal-net Spacings

Let us now consider the special case of adding spacing when the initial cuts don't pass all the way through the plane. And let us try to cover the plane with the series of such cuts. In that case we can get a net of cuts covering the initial molecular plane. In the easiest cases the cuts split the plane to hexagons, so I will refer to this type of cutting the molecular plane as to the *hexagonal net*. The net from Figure 7a consists of hexagons of one-molecule length and will result in a tessellation with the hexagonal clusters of triangular molecules, shown in Figure 7c.

Can we combine several net-cuts at once? Yes, we can and Figures 7d and 7f show one of the examples of such combination. Note that as you take bigger nets, i.e. hexagons with 2-molecule sides and bigger, there will be more variants to position the second or any of the subsequent nets.



(a) **(b)** **(c)**

(d) **(e)** **(f)**

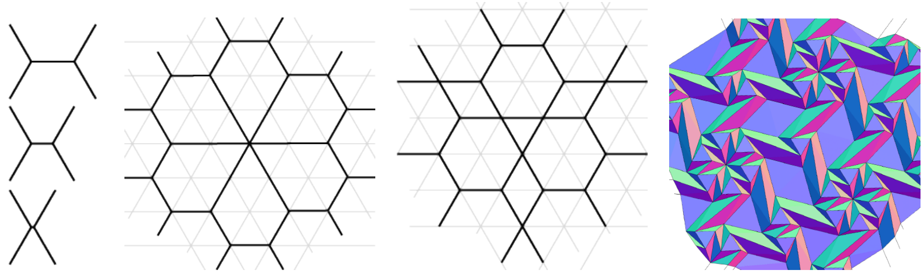
Figure 7: *Hexagonal-net spacings: (a) regular hexagonal net, (b) the crease pattern for spacing* 1 *and Generalized Resch molecule, (c) folded model, opus T-229, (d) two nets aligned (bold and dotted), (e) crease pattern for spacing of* 1*, (f) simulation of the folded model.*

Degenerate Hexagonal-nets

If we consider the length of one side of the hexagon in a net of cuts, we can find out, that by taking a zero length we can arrive to some interesting cases of *degenerate hexagons*, which are not hexagons anymore, but can still be present in hexagonal nets. If we consider the length of one side of a hexagon, we can see that that side should have segments of other sides attached to it at a 120° angle. If we tend the length to zero (see Figure 8a), we come to a degenerate zero-length hexagon side, that turns hexagon to a lesser-side polygon. See Figure 8b for an example of the net with pentagons that play as hexagons and Figure 8c for an example of a hexagons degenerated to a triangle and rhombi. It appears that these cases of nets are possible and produce valid tessellations, see Figure 8d for an example.

Hexagonal-net Spacing Taxonomy?

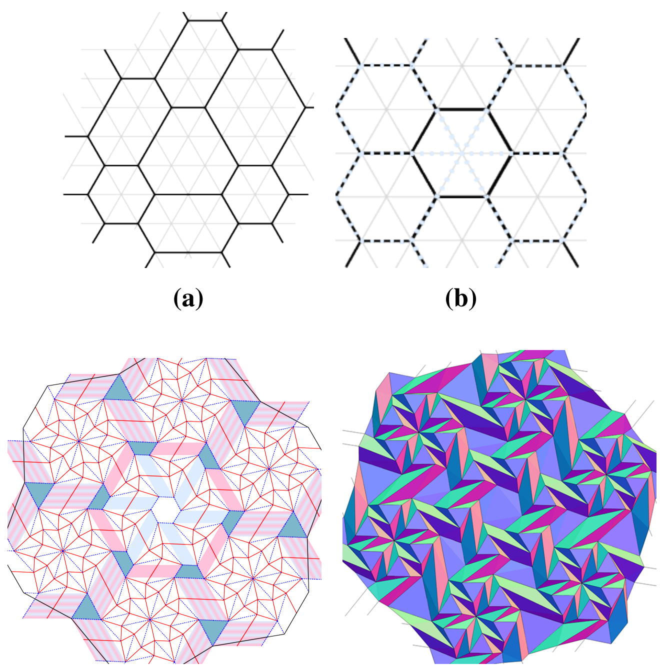
Could we somehow classify the tessellations made with hexagonal nets of cuts? It appears a very complicated if at all doable task. Unlike the straight cuts, the dimensions of new hexagons you add to the existing net follow



(a) (b) (c) (d)

Figure 8: *Degenerate hexagonal-net spacings: (a) let us tend side length to zero, (b) degenerate net example with pentagons instead of hexagons, (c) degenerate example with a triangle and a rhombuses, (d) simulated result of a tessellation with spacings added for the previous cut schema.*

very little rules. Hexagons can be irregular with almost any combinations of side lengths. See Figure 9a for an example of a very irregular net. While two adjacent hexagons in the net touch each other and hence depend on each other, this relationship does not carry on any further. One side of the net may be completely independent from the other side if they are far enough from each other. Even if we consider only the nets that can tessellate an entire space if repeated, there are still too many variants.



(a) (b)

(c) (d)

Figure 9: *Hexagonal-net spacings: (a) an example of an irregular net, (b) combining hexagonal-net cuts(bold black) with straight cuts(bold dotted light) example: we can both prolong the straight cuts to infinity or “adsorb“ them to hexagonal net, doubling its spacing size as we are doing in a crease pattern later, (c) resulting crease pattern with $\frac{1}{2}$ straight spacings adsorbed later to $\frac{1}{2}$ hexagonal spacing, producing spacing of length 1: striped areas represent spacing two types of spacings combined, (d) the resulting tessellation (simulation).*

Straight-cut and Hexagonal-net Combinations

Can we combine hexagonal-net spacing cuts with the straight cuts? It appears that we can. Let us take a regular hexagonal net on molecular plane and add three straight cuts coming intersecting in the middle, see Figure 9b. We can see, that there are segments, where straight-line cuts are coming through the same exact lines as the hexagonal-net cuts. In this case the resulting spacing in those areas will be equal to the sum of the hexagonal-net spacing length and the straight cut length. Every time several different cuts come through the same segment on molecular plane the lengths added with those cuts will be added together. See example in Figure 9d. For the sake of beauty I adsorbed the straight cuts to hexagonal on the border, which resulted in douuble spacing. There is another way to look at this cut combination: we can think of it as of a combination of cuts from Figure 8b with a regular hexagonal net.

Summary and Conclusions

This paper demonstrated the wide range of design techniques that can be used with any spatial triangular twist molecules to create an infinite amount of new shapes. With some adjustments these methods can be used for flat twist tessellations covered extensively in [3].

Acknowledgements

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