Penrose Tiling Arrangements of Traditional Islamic Decagonal Motifs

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Abstract

Decagonal motifs found in the traditional tile designs of Islamic art and architecture can, with a little refinement, be arranged according to the fundamental quasiperiodic structure of the Penrose tiling. A simple tessellation is applied to six traditional motifs to create analogues to the pentagonal Penrose tiles. Substitution rules guiding hierarchical assembly of these Penrose tiling arrangements are then applied, mediated by the tessellation tiles. I then explore how the artist may include hints to the viewer about the mathematical characteristics of Penrose tilings that these designs represent, including their correct arrangement, substitution rules, quasiperiodicity, and other features.

Introduction

The fundamental structure of the Penrose tiling can be decorated with traditional Islamic geometric tile patterns [1][3]. Here I develop a general method for rearranging traditional decagonal motifs into the Penrose tiling framework (Figure 1) by adapting the polygonal method to create elements equivalent to the pentagonal Penrose tiles that can follow the substitution rules via hierarchical assembly. I start with existing polygonal and tessellation analyses of traditional geometric designs [1][10]. I will draw heavily from a specific tessellation method of Majewski [10] to adapt a specific case, found intuitively, to other motifs.

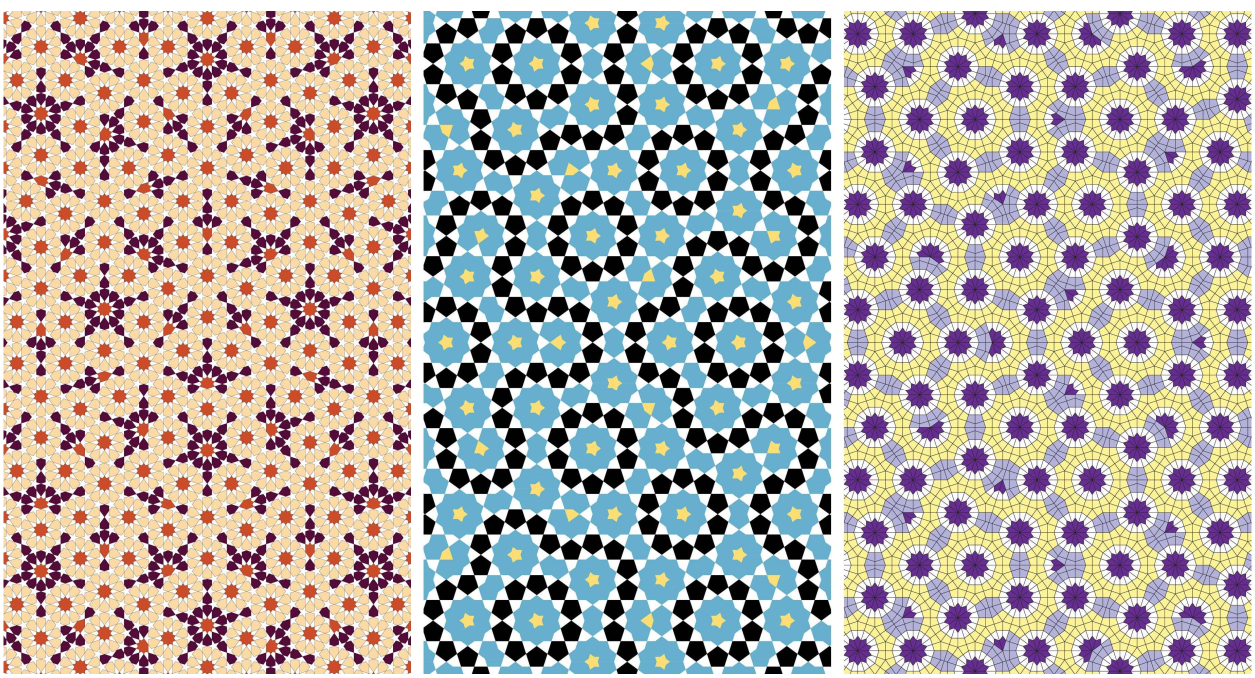

Figure 1: *Traditional Islamic geometric motifs arranged into Penrose tilings. Some motifs will be named as in [10], with the recognition that most motifs appear historically in many different locations and renditions. From left to right: the Rosette motif (named Kukeldash in [10], composed of the "Tond" tile set [2]), Nodir Devon motif, a subset of the "Kond" tile set, and the Decagonal Pandjara motif.*

Substitution Rules and Hierarchical Assembly of the Pentagonal Penrose Tiling

Let us begin by considering the pentagonal Penrose tiling and its substitution rules, shown in Figure 2 [7][8][12]. Of the three standard Penrose tilings (pentagonal P1, kite-and-dart P2, rhomb P3), I am choosing to work with the pentagonal version (P1) in order to engage with its substitution rules as a prescription for a hierarchical assembly process. The Penrose tiling, often defined by its edge matching rules, can be generated by its substitution rules. The Penrose tiling is quasiperiodic and has local isomorphism [13].

In the pentagonal Penrose tiling there are six distinct tiles. I will refer to the three pentagons as P-5, P-3, and P-2, depending on whether they are connected to five, three, or two other pentagons at the edges. For the remaining three tiles, I use the original names as given by Roger Penrose [12]: Star, Boat, and Diamond. A finite selection of the pentagonal Penrose tiling is shown in Figure 2e. In Figure 2b-d,f-h, I show three levels of hierarchical assembly, starting with level 0, the original tiles. Levels 1 and 2 are needed to give us clues in the next section when we look for tiles and rules that can work with a traditional decagonal motif.

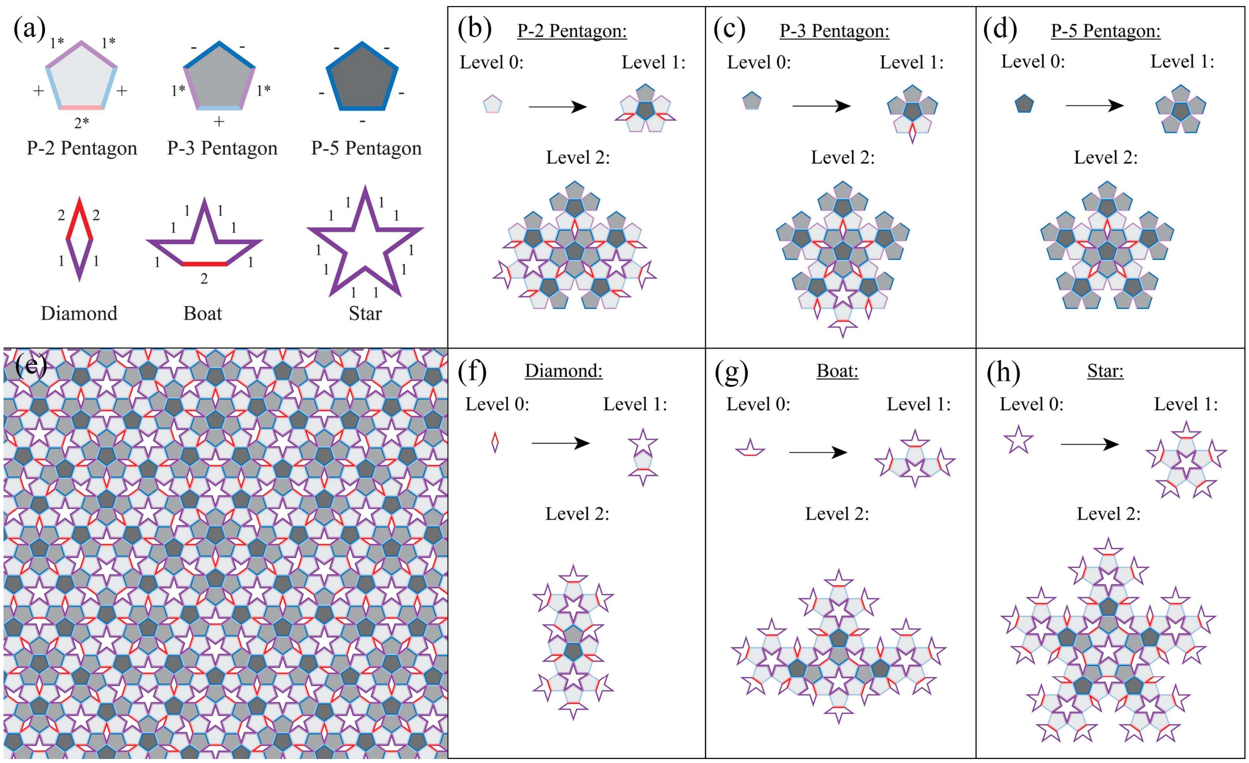

Figure 2: *(a) The set of six distinct pentagonal Penrose tile types. Edge matching rules are indicated by color, where the light and dark versions of the color are a match. These edge colors correspond to the edge labels from [8]. Colors inside the pentagonal tiles help us keep track of their Level 0 identity. (b-d) Substitution rules for the pentagons [8]. (e) A pentagonal Penrose tiling. (f-h) Substitution rules for the Diamond, Boat, and Star tiles [8].*

Here we shall view the substitution rules as guiding a hierarchical assembly process, where all tiles remain fixed in size, and the resulting arrangements of tiles can be treated as ‘supertiles’, following the same substitution rules as the original tiles at the next level in the hierarchy. The set of six tiles with their edge-matching rules constitute Level 0 of the hierarchy. Level 1 is the result of applying the substitution rule. The Level 1 supertiles are functionally equivalent to the Level 0 tiles when applying the substitution rule a second time on these supertiles to produce the Level 2 supertiles. Each supertile obeys the matching rules by construction.

Due to the local isomorphism property of the Penrose tiling, any finite portion of the tiling may be produced by assembling supertiles to a high enough level in the hierarchy such that the finite portion of interest is contained within one or more of the supertiles at that level. Thus, any finite region of a Penrose tiling may be produced via this method, and growth is exponential.

Tessellated Rosette Motifs

I start with the ten-petal Rosette motif (Figure 3), ubiquitous in traditional Islamic design, composed from the "Tond" tile set [2][3]. This tiling only requires three tessellation tiles as analyzed by Majewski [10]. The Rosette motif will be intuitively adapted to the pentagonal Penrose tiles here, to give us a ground-up starting point for the tesselation. This motif has previously been adapted to decorate the Penrose rhombs in [3], where, in the absence of defined Penrose matching rules on the tiles, one could extrapolate to two distinct P3 Penrose arrangements of the rosette motif based on two possible orientations of the thin rhomb. One of those two options yields the Rosette Penrose tiling that we develop next (Figure 1a, Figure 5). One can easily translate tile decorations between the three standard Penrose tilings, P1, P2, and P3. However, the purpose here is to equate pentagonal Penrose tiles to aggregates of whole Islamic tile shapes that can be operated on by the same substitution rules to prescribe a hierarchical assembly process with no overlaps and no gaps at the matching edges of the aggregates.

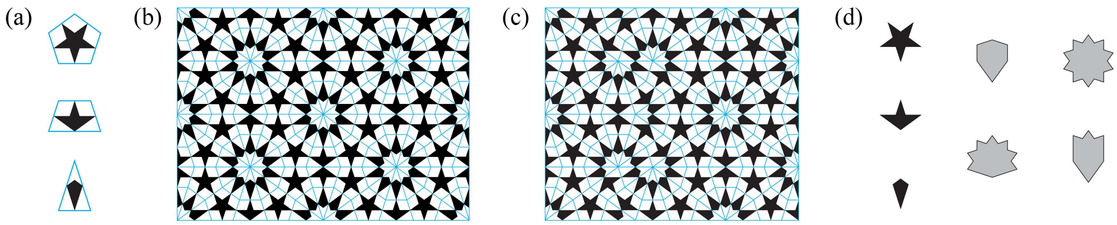

Figure 3: *(a) The three decorated tessellation tiles we will use for our Penrose arrangement. The tessellation by Majewski [10] uses (top to bottom) the Penta tile from the Girih set, and the Cone and Pyra tiles from the Φ-category adapter tiles of [5]. (b) Repeating decagonal pattern of rosette attributed to the Kukeldash Madrasah, with the tessellation from Majewski overlaid [10]. (c) A pattern from the Tomb of I'timād-ud-Daulah in Agra giving us two partial rosette motifs that will help complete the Penrose arrangement. Note that this tiling is also completely tessellated by the same three decorated tessellation tiles as in (b). (d) The set of tiles shapes appearing in these and our Penrose tiling, known as the Tond tile set [2]. The tile shapes in gray appear in the negative spaces between the black tile shapes.*

The polygonal method explained in great depth in Bonner's work [1] and in Majewski's work [10] is key to our ability to adapt hierarchical substitution rules to many motifs. I refer here to the polygons as a tessellation, noting that these three tessellation tiles, shown in Figure 3A, are called Penta, Pyra, and Cone in Erikson's analysis [5]. We will also need the shapes contained in the Sunrise and Shield tiles from Erikson's short tiles category [6], but we will not need to switch to the short tile method of that paper.

Heuristic Discovery of Penrose Substitution Rules for a Rosette Motif

Matching Rosettes to the Pentagonal Penrose Tiles

I take an intuitive, heuristic approach to adapting the Rosette motif to the pentagonal Penrose tiling and its substitution rules. Let us begin working in the visual gestalt of the petals, suns, and other shape elements of the Rosette motif. Later we switch back to the tessellation for its adaptability in applying the result to other decagonal motifs. In the Rosette gestalt, we will encounter shapes that are all part of the "Tond" tile set [2]. I will refer to traditional Islamic tile shapes as "tile shapes", Penrose and tessellation tiles as "tiles", and finite arrangements of tile shapes that equate to Penrose tiles as "aggregates."

Given the decagonal symmetry of a full rosette, it makes sense to place its symmetry center over the 5-fold symmetry center of the pentagonal tiles P-5, P-3, and P-2. We can use color to keep track of which tiles correspond to each other. We will consider small aggregates of tile shapes to be equivalent to a Level 0 Penrose tile, meaning that there is no further breakdown possible. We start with the Level 0 P-5 pentagon. To replicate its connectivity to five other pentagons at Level 1, place 5 more rosettes symmetrically around the central rosette to guess at the P-5 Level 1 construction (Figure 4). We now consider the gaps that are created. It is clear the small star shape of the Rosette motif can be used to fill the small gaps between rosettes, so we start by surrounding the P-5-equivalent rosette with these stars (Figure 4A). They are like an edge glue or edge shape, which is a way of interpreting matching rules for edges. The P-5 pentagon is symmetric, so we give it a symmetric corona of these star motifs.

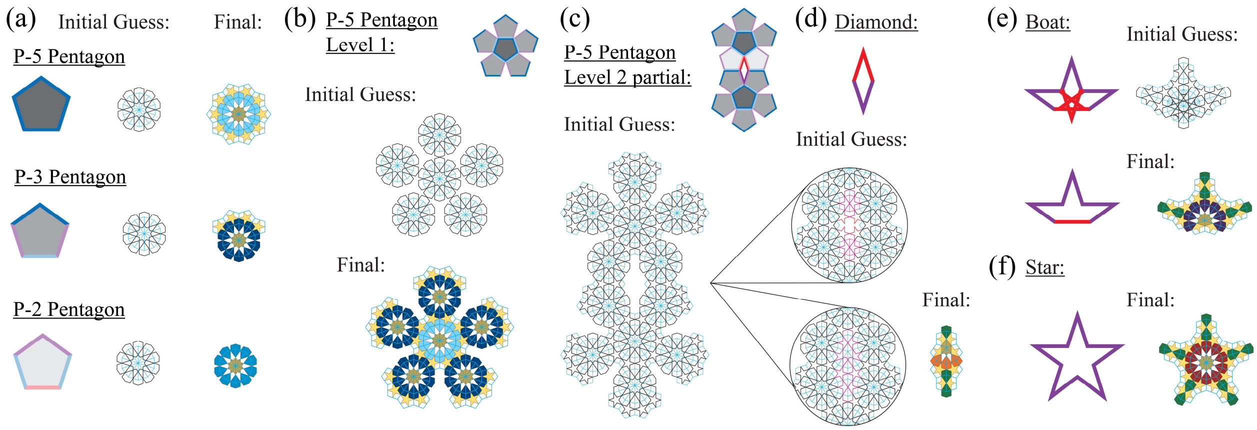

Figure 4: *The process of determining the Rosette tile aggregates equivalent to the pentagonal Penrose tile set.*

Next, we note that the P-3 pentagon has two edges identical to the P-5 pentagon, so we give those edges the star shapes as well. The other edges in P-3 and P-2 pentagons we leave devoid of star shapes, guessing that this will allow matches without overlap or gaps. As we proceed, we must make guesses and revise if we find overlaps or voids after applying the substitution rules in hierarchical assembly.

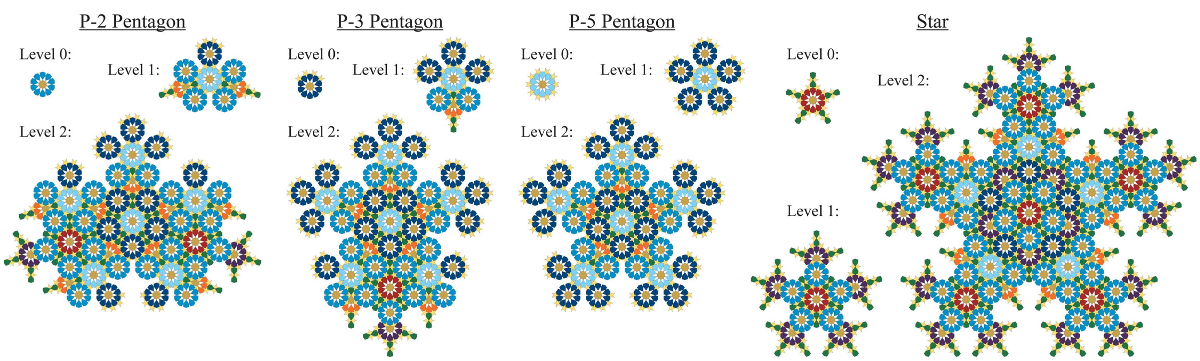

Figure 5: *Substitution rules applied over Levels 1 and 2 verify that our tile aggregate choices work with the pentagonal Penrose substitution rules. These four Level 2 supertiles are sufficient verification, as the Star repeats the same interactions as the Diamond and Boat.*

The P-5 pentagon has simple and symmetric interactions, so we move on to guess at the construction of the Level 2 P-5 pentagon for more clues (Figure 4b). We need only attach the Level 1 P-3 pentagon to the Level 1 P-5 pentagon, creating another gap to fill, where the Diamond tiles mediates the interaction (Figure 4C). We look to other traditional patterns for motifs and we find the three-petalled rosette with the Shield-shaped center fits the space (Figure 3C). By taking cues again from the existing edge glues, we decide which way to orient the petals. Now we have an equivalent to the pentagonal Penrose Diamond tile. However, we will find later that we can also choose the other orientation and still have success with the substitution rules. In fact, the interior design of any Level 0 aggregate can be modified, a quality that we will use to solve awkward design issues that arise.

Next, we overlap three pentagonal Penrose Diamond equivalents to get the same edges as the Boat and the Star, and look to see how it should be modified (Figure 4E). The overlap strongly suggests the structure we choose here for these tiles, where we place the Sunrise tile shape at the center of the overlapping region and utilize the Cone tessellation tiles at the base. Given the makeup of the Diamond-equivalent aggregate, we also maximally surround the Boat and Star equivalent aggregates with the small star shape motifs. At this stage these are all guesses. We shall verify that they function as desired by assembling all the Level-1 and Level 2 supertiles in Figure 5.

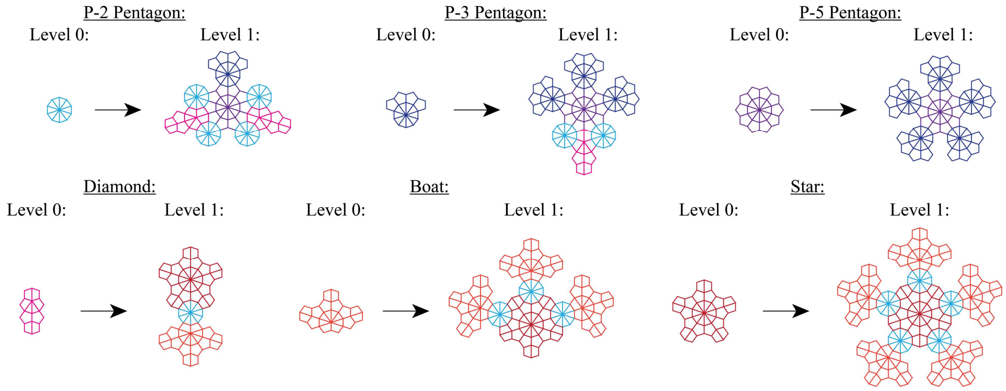

Figure 6: *Substitution Rules for Tessellation Equivalents to Pentagonal Penrose Tiles. We color the Level 0 tessellation aggregates to help the reader keep track. Tessellation aggregate colors are unrelated to pentagonal Penrose tile colors shown previously. The substitution follows the same rule as in Figure 2.*

Result: Tile Aggregates Analogous to the Pentagonal Penrose Tiles

Now follows a process of applying the substitution rules to build all the Level 2 supertiles after the Level 1 tiles are completed. We verify in Figure 5 that our proposed tile aggregate set is indeed consistent with the substitution rules over two levels. It is now intuitively clear that we have a working tile set to produce Penrose patterns. We find that the three tessellation tile shapes are sufficient to completely tile the Penrose tiling, and that the substitution rules work perfectly with no gaps and no overlaps. We have found tile aggregates of these three tessellation tiles that act as analogs to the six pentagonal Penrose tilings under the substitution rules.

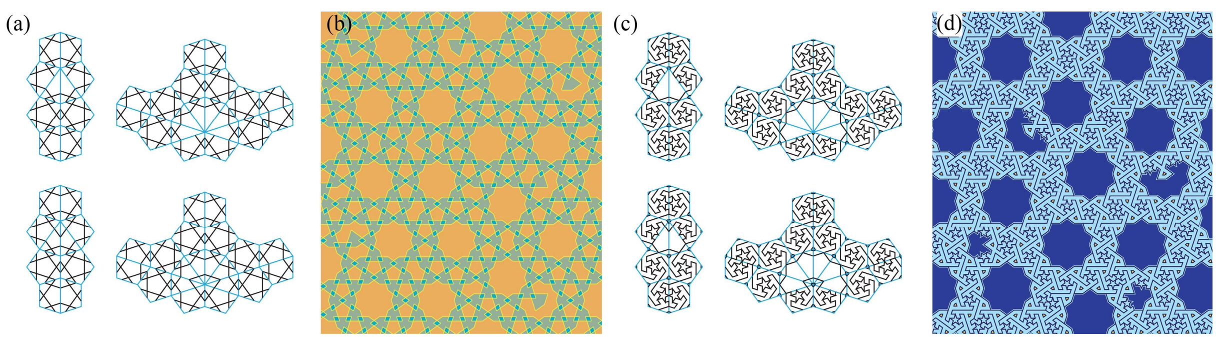

Figure 7: *Details of solutions to design challenges in the partial decagons. (a) Missing element of Tan Sahid solved with additional Pyra tiles. (b) Tan Sahid motif as a Penrose tiling, (c) Awkward elements of the Yazd motif solved with Pyra tiles and a mirror image Cone tile. (d) Yazd motif as Penrose tiling.*

Decagonal Tiling Motifs Embedded in Penrose Tilings via Their Tessellations

Applying the Tessellation Substitution Rules to Other Decagonal Motifs

The mapping between our heuristic Rosette Penrose and the Majewski tessellation gives us tessellation tile aggregates that can carry other tiles shapes (Figure 6). We can use the substitution rules we found for the Majewski tessellation tiles to turn any decagonal tiling with a Majewski tessellation into a Penrose arrangement. This is not the only system of tile aggregates that could work this way, nor the only tessellation or substitution rule that could be applied. But, by making decorated polygon aggregates that are analogous to the pentagonal Penrose tiling set, there is an immediate aesthetic connection between the decagonal patterns and the Penrose structure that can benefit the artist wishing to create these tilings. Other substitution rules for the three-tile Majewski tessellation set can be inferred from this work, but I will not follow this direction here.

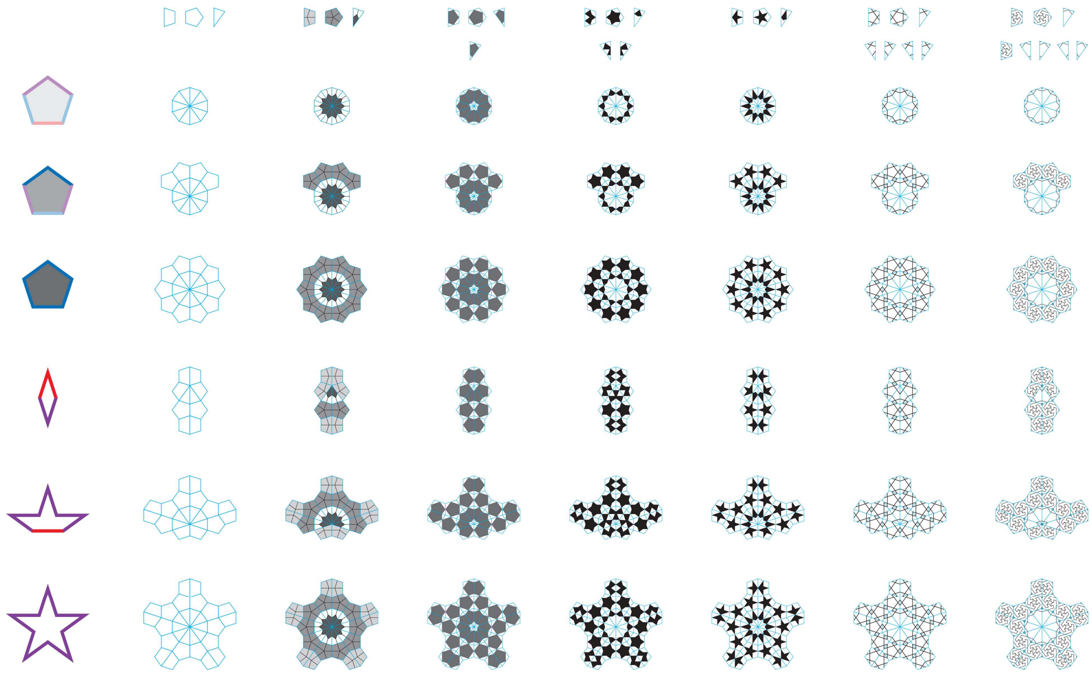

Figure 8: *Main Result: Tile shape aggregates equivalent to the pentagonal Penrose tiles, adapted from traditional motifs via the tessellation tiles. Left to right the motifs are: pentagonal Penrose, Tessellation, Decagonal Pandjara, Nodir Devon, Persian, Rosette, Tan Sahid, and Yazd.*

Solving Design Dilemmas

Majewski's analysis [10][11] includes some rules of thumb to solve design dilemmas in cases of unexpected adjacency of certain tiles. Here, we may follow these guidelines and our own aesthetic to find suitable solutions to those design dilemmas in Figure 7. Having addressed these details we have a full set of tessellated tile aggregates, analogous to the Level 0 pentagonal Penrose tiles that carry traditional decagonal motifs into the Penrose arrangement (Figure 8).

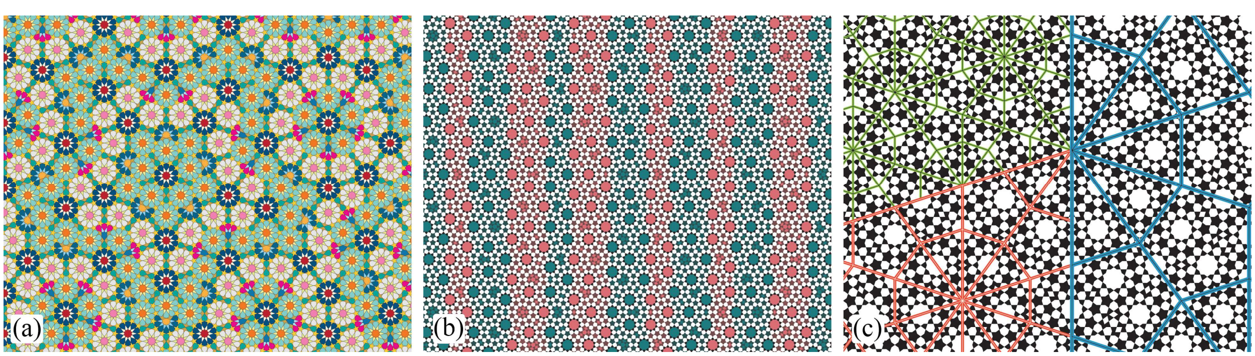

Figure 9: *(a) Extensive tiling and coloration of the Rosette motifs reveal recursive levels and features of the Penrose tiling. We also see that the Rosette design can be reversed within the Diamond-equivalent aggregate, as mentioned above. (b) Ammann bars in one direction highlighted with color in the Persian Penrose tiling, revealing the Fibonacci sequence of thick and thin lines. (c) Multiple levels of tessellation of the Persian Penrose tiling suggest the recursive structure and tessellation analysis of the pattern.*

Artistic Interventions: How the Artist Reveals Knowledge of the Penrose Tiling in the Design

Revealing Features and Knowledge About the Penrose Tiling

Traditional Islamic geometric tilings reveal a sophisticated understanding of geometry held by their designers. In general, how can Penrose arrangements of traditional patterns be presented to inform the viewer about the level of our understanding of the Penrose tiling? Bold claims of quasiperiodicity in traditional Islamic patterns, such as the Darb-e Imam shrine in Isfahan, have been made [9] and refuted [4]. Thus, it is worth asking, what can the designer do to convince the viewer about the designer's knowledge of the mathematics behind the Penrose tiling?

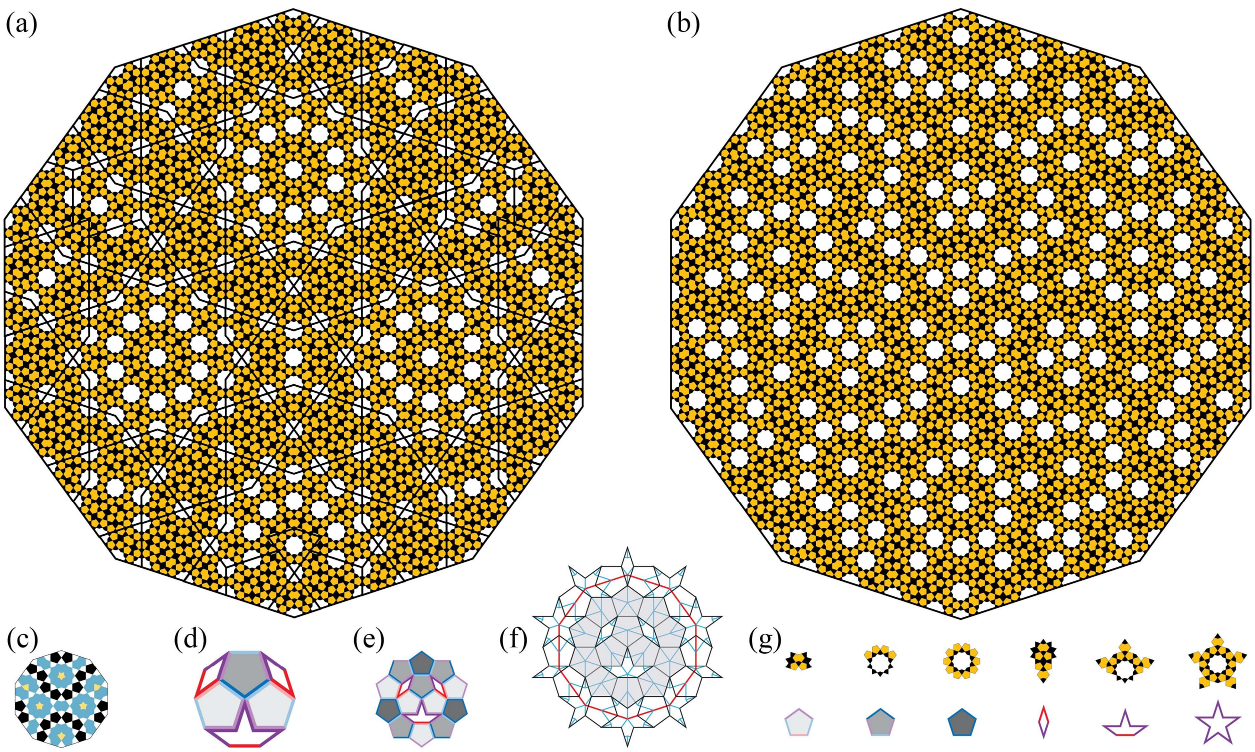

Figure 10: *(a) Penrose arrangement of Kond tile shapes from the Darb-e Imam shrine. (b) Alternative Penrose arrangement of Kond tile shapes. (c) Nodir Devon tile arrangement guiding (a) and equivalent to (d). (d) The center of the pentagonal cartwheel motif derived in (f). (e) Pentagonal cartwheel motif. (f) The term "cartwheel" was coined by John Conway to describe the arrangement of kite and dart Penrose tiles (cyan) inside the red outlined regular decagon shown here (rotated 180° from [7]). From the translation into the pentagonal Penrose tiling [8], I adopt the inner wheel of pentagons and its interior as the pentagonal cartwheel (gray). (g) Pentagonal Penrose equivalent aggregates producing (b).*

One of the most effective ways to convey precise knowledge of the Penrose tiling to the viewer is to create very large tilings and to add coloration that assists the eye in seeing the fractal nature, scaling, symmetry, and aperiodicity (Figure 9a). One must understand the construction very well to achieve this and it will be evident to the viewer. I list here many ways to demonstrate advanced knowledge of the mathematics of the Penrose tiling: 1) show a large area, with coloration that reveals structure (Figure 9a), 2) show multiple recursive levels of the design (Figure 9c), 3) highlight features such as Ammann bars (Figure 9b), cartwheels (Figure 10), and centers of 5-fold symmetry, 4) highlight or feature the substitution rules, 5) show how one Penrose tiling maps to another, 6) include decorations within the framework that contain words, definitions, proofs, geometric constructions, the Golden ratio, Fibonacci series or any other related mathematical details. Traditional Islamic geometric tilings often include design elements and calligraphy within tile shapes or spaces created within a tiling. There is ample room within this framework to match traditional design elements with present-day mathematical knowledge.

Application to the Darb-e Imam Shrine Decagonal Pattern

The Darb-e Imam shrine tiles may be arranged to strongly suggest quasiperiodic Penrose structure. The Nodir Devon tiling in this paper has the same tile shapes as the higher-level tiling in the Darb-e Imam shrine tiling. I apply the Nodir Devon Penrose tiling arrangement at this level (Figure 10a). I chose the smallest strongly characteristic arrangement within the Penrose tiling: the cartwheel motif (Figure 10c-f)[7], in this case, just the center of it (Figure 10d). Here the smaller tile shapes play no role in Figure 10a other than decoration of the Nodir Devon tile shapes. For another approach, Figure 10b, I show one possible mapping of the Kond tile shape set to pentagonal Penrose equivalents, deviating slightly from the tessellation used previously in this paper. This can be done in many ways, but I chose a way that replicates the circle of ten decagons seen in the original. With this approach, the Penrose structure manifests in new ways, different from the traditional Darb-e Imam tiling (Figure 10b).

Summary and Conclusions

To conclude, we have utilized the polygonal technique with the three Majewski tessellation tiles (Penta, Cone, and Pyra) to adapt a set of traditional Islamic decagonal motifs into the Penrose tiling structure by creating analogues to the pentagonal Penrose tile set. We have interpreted the substitution rules of the Penrose tiling as a hierarchical assembly process of tile shape aggregates composed from these tessellations. Finally, we have judiciously intervened in the resulting tilings to highlight structural features of the Penrose tiling such as recursive and quasiperiodic structure, cartwheel motifs, Ammann bars, and multiple levels of tessellation to convey mathematical knowledge to the viewer.

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