Ptolemy, the Regular Heptagon, and Quasiperiodic Tilings

Peter Stampfli[1] and Theo P. Schaad[2]

[1] Rue de Lausanne 1, 1580 Avenches, Switzerland; pestampf@gmail.com
[2] 2704 38th Ave SW, Seattle, WA 98126, USA; theoschaad@gmail.com

Abstract

We examine the substitution method for creating seven-fold quasiperiodic tilings with rhombi. From Ptolemy's theorem, we obtain relations between the sides of the regular heptagon and its diagonals. This then gives us the length of diagonals of the rhombi and defines the possible inflation ratios. For a given inflation ratio, we obtain the numbers of the various small rhombi required to substitute the entire inflated rhombus tile as well as the rhombi at its border. For mirror symmetric substitutions, we also get the rhombus tiles at the mirror axis. A quasiperiodic tiling of seven-fold rotational symmetry is presented and examined with respect to these results. This approach could be used to create tilings of five-fold and other rotational symmetries.

Introduction

We consider quasiperiodic tilings of seven-fold rotational symmetry using three different rhombi as tiles. They result from an iterative substitution method. It increases the size of rhombi by an inflation factor and replaces them by small rhombi of original size. It can be very difficult to find substitution rules that can be repeated consistently and give an infinite tiling. We use Ptolemy's theorem and the geometry of the regular heptagon to find relations between an inflation factor and the small rhombi lying on the side or diagonal of an inflated rhombus. The numbers of various rhombi replacing an inflated one are also obtained. This makes it much easier to find valid substitution rules.

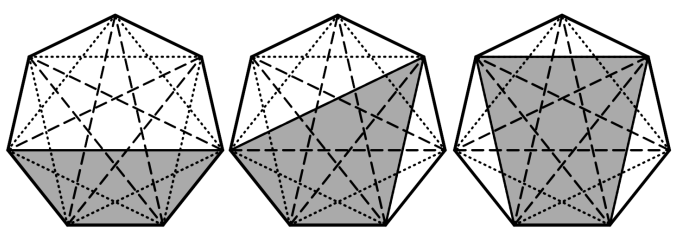

Figure 1: *Regular heptagons with cyclic quadrilaterals, as used in applying Ptolemy's theorem.*

Ptolemy's Theorem and the Regular Heptagon

The corner angles of a regular heptagon are $5\pi/7$ and the angle between its diagonals and sides are integer multiples of $\pi/7$ because the sum of angles of an Euclidean triangle is equal to π. The regular heptagon has diagonals of two different lengths. We let its sides define the unit length. We then use φ for the length of the short diagonals (dotted lines in Figure 1) and ρ for the long ones (dashed lines). From Ptolemy's theorem, we get important relations between φ and ρ.

Cyclic quadrilaterals have all four corners on the same circle. Ptolemy's theorem [11] states that the product of the diagonals of such a quadrilateral is equal to the sum of the products of opposite sides. This is a generalization of Pythagoras' theorem.

Any four corners of a regular heptagon define a cyclic quadrilateral. The one on the left of Figure 1 has a long diagonal of the heptagon as one of its sides together with three sides of the heptagon. Its diagonals are short diagonals of the heptagon. Ptolemy's theorem gives us thus

$$\varphi^2 = \rho + 1 \tag{1}$$

The quadrilateral at the center of Figure 1 has a long and a short diagonal as sides. Both are opposite to sides of the heptagon. Its diagonals are a long and a short diagonal. Thus

$$\varphi\,\rho = \varphi + \rho \tag{2}$$

Finally, three sides of the third quadrilateral are short diagonals of the heptagon and the fourth is a side of the heptagon. Its two diagonals are long diagonals of the heptagon. This gives

$$\rho^2 = \varphi + \varphi^2 = \varphi + \rho + 1 \tag{3}$$

These equations are similar to the definition of the golden ratio [10]. From (1) and (2) we get a third order polynomial equation $\varphi^3 - \varphi^2 - 2\varphi + 1 = 0$. Its solutions require cubic roots. This shows that a regular heptagon cannot be constructed using only compass and straightedge [4]. Basic trigonometry gives the solution that is relevant for the heptagon

$$\varphi = 2\cos(\pi/7) \approx 1.802 \tag{4}$$

Then, using equation (1) we get

$$\rho = \varphi^2 - 1 \approx 2.247 \tag{5}$$

Dividing both sides of equation (2) by φ gives the quotient

$$\rho/\varphi = \rho - 1 \tag{6}$$

Similarly, from equations (1 – 3) we obtain

$$\varphi/\rho = \varphi - 1 \tag{7}$$

$$1/\varphi = \varphi - \rho/\varphi = \varphi + 1 - \rho \tag{8}$$

and

$$1/\rho = \rho - 1 - \varphi/\rho = \rho - \varphi \tag{9}$$

Thus for φ and ρ multiplication and division become addition and subtraction [10]. The set H of linear combinations of the unit, φ, and ρ with integer coefficients is a commutative ring, similar to Gaussian integers. The equations above define the multiplication of its elements and some nontrivial divisions. As we shall see in the following, the inflation factors of most seven-fold tilings are elements of H.

Inflation Factors and Substitutions at Borders

Rhombic quasiperiodic tilings with seven-fold rotational symmetry use three different rhombi A, B, and C with acute angles of $\pi/7$, $2\pi/7$, and $3\pi/7$, having sides of unit length. We now determine the relations between the diagonals of the heptagon and diagonals of these rhombi. We use d_X and D_X to refer to the short and long diagonals of rhombus X, respectively. Three corners of a rhombus with acute angle of $\pi/7$ can be matched to corners of the heptagon, see left of Figure 2. Its sides are the long diagonals ρ of the heptagon, and its short diagonal is a heptagon side of unit length. Scaling to get rhombus sides of unit length and using equation (9) gives us for the short diagonal d_A of a rhombus A

$$d_A = 1/\rho = \rho - \varphi \approx 0.445 \tag{10}$$

From the center of Figure 2 we get the long diagonal D_B of rhombus B as

$$D_B = \varphi \approx 1.802 \tag{11}$$

The short diagonal d_C of rhombus C with acute angle $3\pi/7$ results from a similar rhombus with sides of length φ and diagonal ρ, see right of Figure 2, and using equation (6)

$$d_C = \rho/\varphi = \rho - 1 \approx 1.247 \tag{12}$$

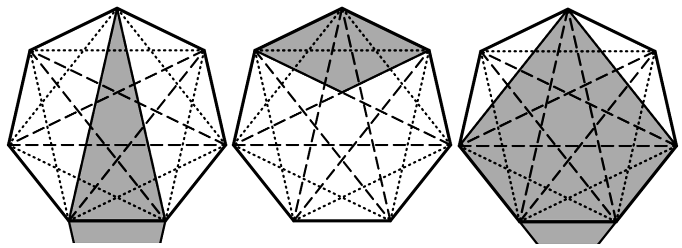

Figure 2: *Rhombi, as used in seven-fold quasiperiodic tilings, fitted to the regular heptagon.*

These three diagonals are elements of the set H. We impose the constraint that the small rhombi at the border of an inflated rhombus have either a side or a diagonal on the border. This results in particularly attractive tilings with centers of seven-fold rotational symmetry. The inflation ratio r is a sum of the length of diagonals and sides of rhombi. Thus it is an element of H and can be written as

$$r(h, k, j) = h + k\varphi + j\rho \tag{13}$$

where h, k and j are integers. Since φ, ρ, and $\varphi/\rho = \rho - 1$ (equation 7) are irrational numbers, each choice for the three integers gives a unique inflation ratio r. Note that for a given r many different border substitutions are possible. Because $\rho = d_A + D_B = d_C + 1$ we can exchange an A and B rhombus pair with diagonals d_A and D_B along the border of an inflated tile with a C rhombus with diagonal d_C and a side of another rhombus. This yields many different seven-fold tilings, particularly for large inflation factors [5]. We present only a few examples and use equation (13) for the inflation factor.

Ludwig Danzer [1] used triangles to create a tiling with seven-fold rotational symmetry and inflation ratio $r = 1 + \varphi \approx 2.802$. The triangles are obviously halves of the rhombi A, B, and C. Rhombi A and B are cut along the short diagonal and B is cut at the long one. Thus, this tiling has tiles with four different edge lengths: 1, d_A, D_B, and d_C, which is quite special. Using triangle tiles instead of rhombi one has more possibilities for creating new tilings. In particular, it is not possible to make a rhombus tiling [5] with such a small inflation ratio.

Chaim Goodman-Strauss [2] created a rhombus tiling with $r = 2 + \varphi \approx 3.802$. However, some rhombi at the border of inflated tiles do not have their side or a diagonal on the border. Thus there is no center of seven-fold rotational symmetry and the tiling has a disordered appearance. It is interesting that this tiling can be generalized to any odd n-fold order.

Joshua Socolar's tiling [7] has $r = 2 + 2D_B + d_c = 1 + 2\varphi + \rho \approx 6.851$. The inflated rhombus tiles have B type rhombi at each corner, with only one exception. Thus many centers of local seven-fold rotational symmetry with seven B rhombi arise.

Alexey Madison [3] proposed a rhombus tiling with $r = 1 + \varphi + \rho \approx 5.049$. It is particularly complicated because it requires nine different substitution rules. However, it has several different centers of seven-fold rotational symmetry, which makes it very attractive. Theo Schaad [6] created another tiling with the same inflation ratio, but which requires only seven substitution rules, see Figure 3. As part of a series of tilings, it has the name "#5".

Substitutions at Diagonals

Note that the substitutions of tiling "#5" shown in Figure 3 are mirror symmetric at the short diagonal for A and C rhombi and at the long one for B rhombi. Thus we now determine the substitution at these diagonals. Given an inflation ratio r we can calculate the length of diagonals of inflated rhombus tilings. Using the product relations (1 – 3) we get again linear combinations of 1, φ and ρ. For the small diagonal of an inflated A rhombus:

$$d_{\mathbf{A}} = r\, d_A = j - k + (k - h)\varphi + h\rho \tag{14}$$

Similarly for the other rhombi

$$D_{\mathbf{B}} = r\, D_B = k + (h + j)\varphi + (k + j)\rho \tag{15}$$

and

$$d_{\mathbf{C}} = r\, d_C = j - h + j\varphi + (h + k)\rho \tag{16}$$

These equations also apply to the base of equilateral triangles as used by Danzer [1]. The inflation ratio of his tiling is $r = 1 + \varphi$, thus $h = k = 1$ and $j = 0$. This gives for the base of inflated triangles $d_{\mathbf{A}} = \rho - 1 = d_C$, $D_{\mathbf{B}} = 1 + \varphi + \rho = 2 + d_C + D_B$ and $d_{\mathbf{C}} = 2\rho - 1 = 2d_C + 1$, as is actually the case.

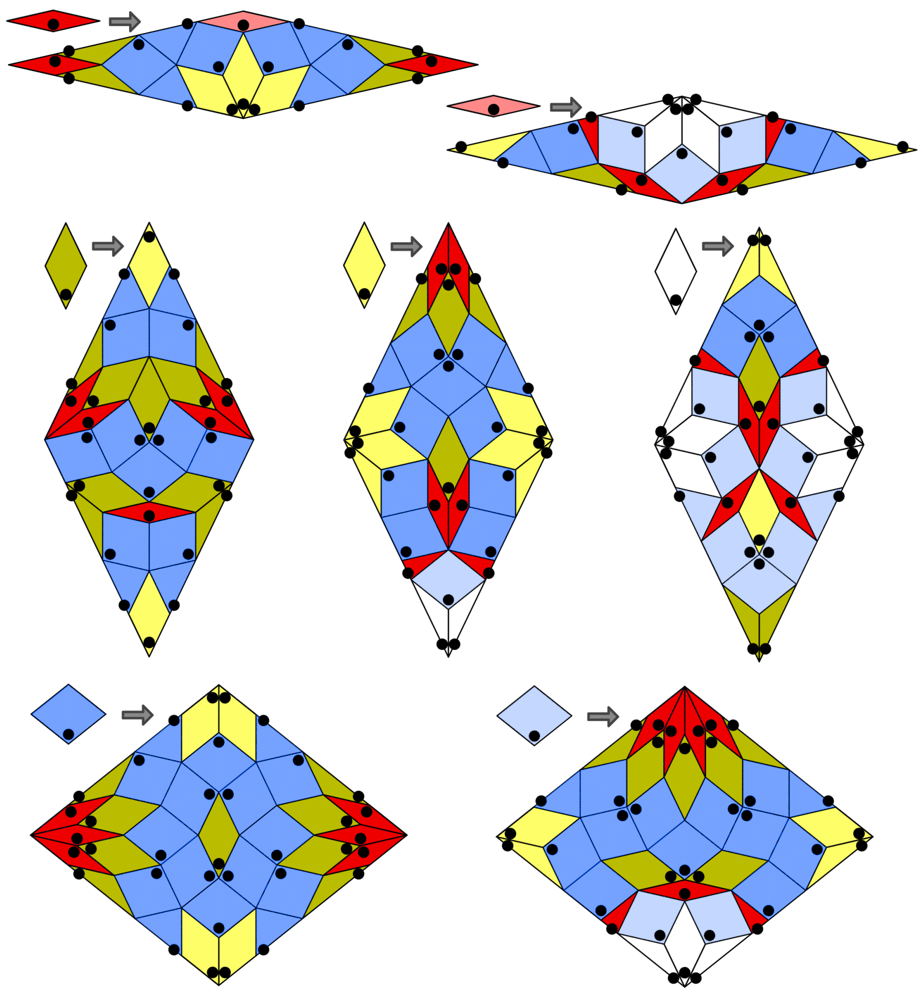

Figure 3: *Substitutions used for Theo Schaad's tiling "#5". The dots indicate the orientation of rhombi.*

Areas and Their Substitutions

We want to determine the number of small tiles required to fill the inflated ones using the same method as for 12-fold tilings [9]. The area of rhombus A with sides of unit length and acute angle $\pi/7$ is

$$A = \sin(\pi/7) \tag{17}$$

Similarly for rhombus B, using trigonometric double angle formulas and equation (4)

$$B = \sin(2\pi/7) = 2\cos(\pi/7)\sin(\pi/7) = \varphi A \tag{18}$$

Rhombus C has an obtuse angle of $4\pi/7$ opposite to its long diagonal, which is thus of length $D_C = 2\sin(2\pi/7)$. The area results from the product of its two diagonals as

$$C = d_C\, D_C/2 = (\rho - 1)\sin(2\pi/7) = (\rho - 1)B = (\rho - 1)\,\varphi A = \rho A \tag{19}$$

using equations (12) and (2). Inflating the lengths of a tile by a factor of r increases its area by

$$r^2 = (h + k\varphi + j\rho)^2 = h^2 + k^2 + j^2 + (j^2 + 2hk + 2jk)\varphi + (k^2 + j^2 + 2hj + 2jk)\rho \tag{20}$$

where we have used equations (1 – 3), and (13). It is convenient to rewrite this result as

$$r^2 = a + b\,\varphi + c\rho \tag{21}$$

where $a = h^2 + k^2 + j^2$, $b = j^2 + 2hk + 2jk$, and $c = k^2 + j^2 + 2hj + 2jk$. Applying this to the A rhombus and using equations (17 – 19) gives the surface $\mathbf{A}$ of the inflated rhombus

$$\mathbf{A} = r^2 A = aA + bB + cC \tag{22}$$

Thus, the number of A rhombi in an inflated $\mathbf{A}$ rhombus is given by a because φ, ρ, and φ/ρ are irrational numbers. Also, the number of B rhombi is b and the number of C rhombi is c. We get similar results for B and C rhombi:

$$\mathbf{B} = r^2 B = bA + (a + c)B + (b + c)C \tag{23}$$

and

$$\mathbf{C} = r^2 C = cA + (b + c)B + (a + b + c)C \tag{24}$$

Thus we know how many rhombus tiles we have to use in each substitution rule [6]. In particular, these numbers are independent of the border substitutions, if we correctly count halves of rhombi, see Figure 3.

Another Set of Diagonals and Inflation Ratios

We determine the length of the second diagonals of the rhombi to get additional inflation factors and substitution rules. The short diagonal of a B rhombus lies opposite to its acute angle of $2\pi/7$, thus its length is $d_B = 2\sin(\pi/7)$. Further, from $d_B^2 = 4(1 - \cos^2(\pi/7)) = 4 - \varphi^2$ and equation (1) we get

$$d_B = \sqrt{3 - \rho} \approx 0.868 \tag{25}$$

Knowing the surfaces of the rhombi, which are half the product of their diagonals, we can get the long diagonals of rhombi A and C. Thus, using equations (10, 11, 18)

$$D_A = 2A/d_A = d_B D_B/(\varphi d_A) = \rho\, d_B \approx 1.950 \tag{26}$$

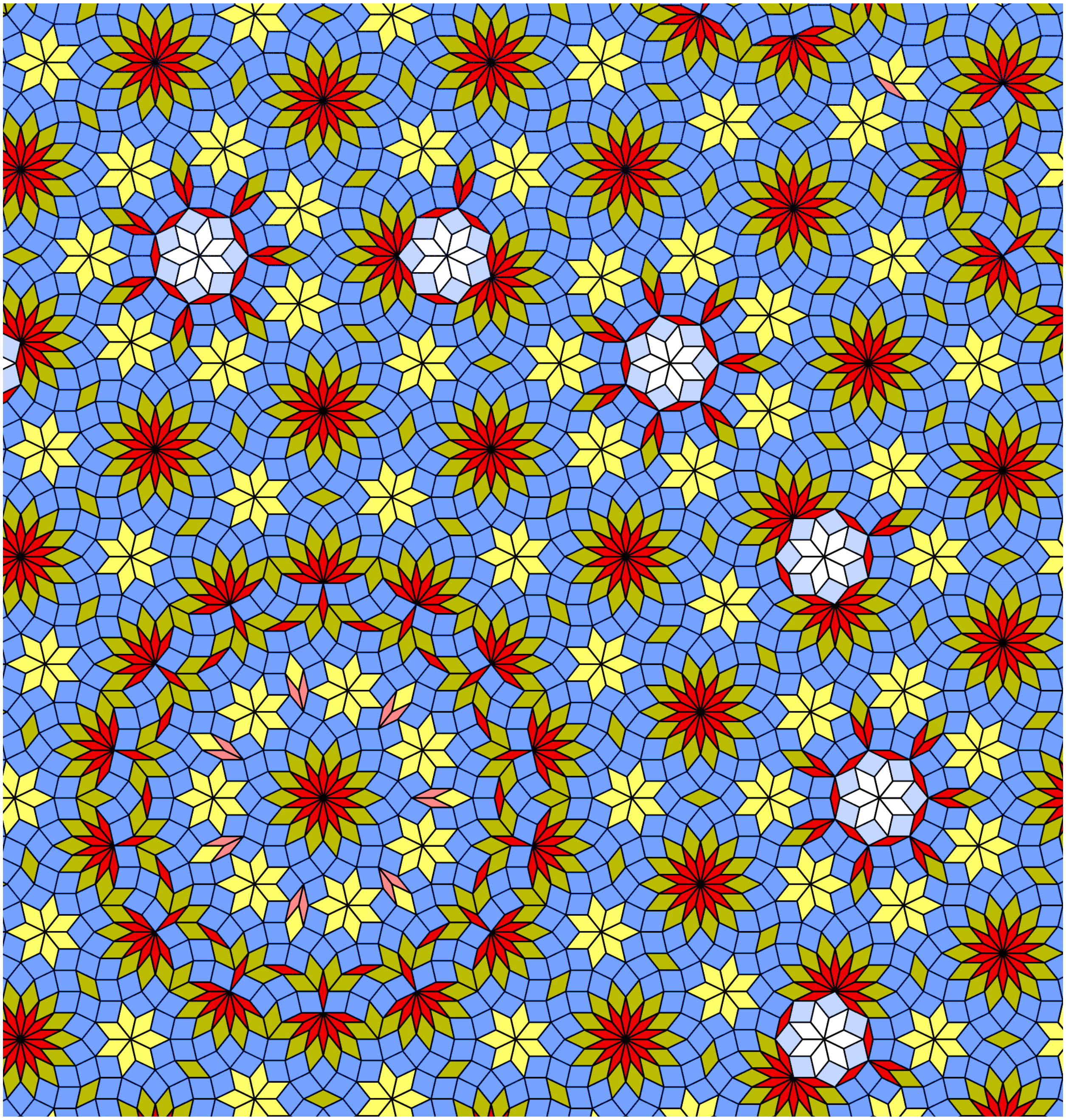

Figure 4: *A patch of Theo Schaad's tiling "#5" generated by a browser app [8]. A center of seven-fold rotational symmetry lies at the lower left.*

and using equations (19, 12, 6)

$$D_C = (\rho - 1)d_B D_B / d_C = \varphi\, d_B \approx 1.564 \tag{27}$$

We can do as before and determine the length of corresponding diagonals of inflated tiles which results in substitution rules at these diagonals.

Because d_B is not an element of H, due to the square root of equation (25), we get a second set of inflation factors

$$s(h, k, j) = (h + k\varphi + j\rho)d_B \tag{28}$$

which could result in different seven-fold tilings. Note that s is not in H, whereas s^2 is an element of H as $d_B^2 \in H$, see equation (25). To get a tiling with centers of seven-fold rotational symmetry all rhombi on the border of inflated tiles should have their diagonal on this border. It is an open question if this is possible. However, this is not the case in the minimal rhombus tiling created by Theo Schaad [5], which has an inflation factor $r = D_A$ and which has a disordered appearance. Yet, he could put together patches of the tiling to get decorative rosettes.

On the Tiling "#5"

This tiling [5] has an inflation ratio $r = 1 + \varphi + \rho = 1 + d_A + 2D_B = 2 + D_B + d_C$ with two compositions. Many different substitutions are possible at the border of an inflated tile as each unit length can be the side of any of the three rhombi. You can see some substitutions and how tiles fit together in Figure 3. Theo Schaad only considered substitution rules that are mirror symmetric at the short diagonal for A and C rhombi and at the long diagonal for B rhombi. This makes it easier to match halves of rhombi across the borders as these diagonals can lie on the border of inflated tiles. This choice also strongly reduces the number of substitutions.

From equation (22) we get that the substitution of an inflated A rhombus has three small A rhombi, five B rhombi, and six C rhombi. Equation (14) for the length of its inflated short diagonal gives $d_{\mathbf{A}} = \rho = d_A + D_B = 1 + d_C$. This shows that there are two different symmetric substitution at this diagonal and that it is not possible to have mirror symmetry at both diagonals, see the top of Figure 3. Seven mirror symmetric substitutions have been found [6]. Similarly, an inflated B rhombus gets substituted by five A, nine B, and eleven C rhombi, see equation (23). The length of its long diagonal is $D_{\mathbf{B}} = 1 + 2\varphi + 2\rho$, which allows for many different combinations, see equation (15). Twelve mirror symmetric substitutions have been found [6]. An inflated C rhombus has six A, eleven B, and fourteen C rhombi. There are nine substitution rules with bilateral symmetry at the short diagonal [6]. One particular choice of these rules gives Madison's tiling [3].

Theo Schaad [5] chose first the substitution rule of the inflated A rhombus at the top left of Figure 3 because it has itself small A rhombi at its acute corners. Thus, repeated inflation and substitution of a rosette of fourteen such rhombi would reproduce this rosette and add layers of a quasiperiodic tiling [9]. We see that this rhombus has a common border with a B rhombus. We get a good fit with the substitution rule at the center left of Figure 3. This one in turn borders with a C rhombus with the substitution rule at the lower left. Continuing in this way a complete set of seven substitution rules has been found. A patch of the resulting tiling is shown in Figure 4. You can explore the tiling with a browser app [8], which can also generate other tilings and fractals.

Besides the rosettes of thin A rhombi, there are also rosettes of the larger B rhombi. Their inflation and substitution also gives centers of seven-fold rotational symmetry. Each of the three B rhombi in Figure 3 thus creates one of the distinct patterns shown in Figure 5. From Figure 3 we see that inflation and substitution exchanges the small rhombi at the corners of the inflated tiles. This thus exchanges cyclically the corresponding three patches of seven-fold rotational symmetry.

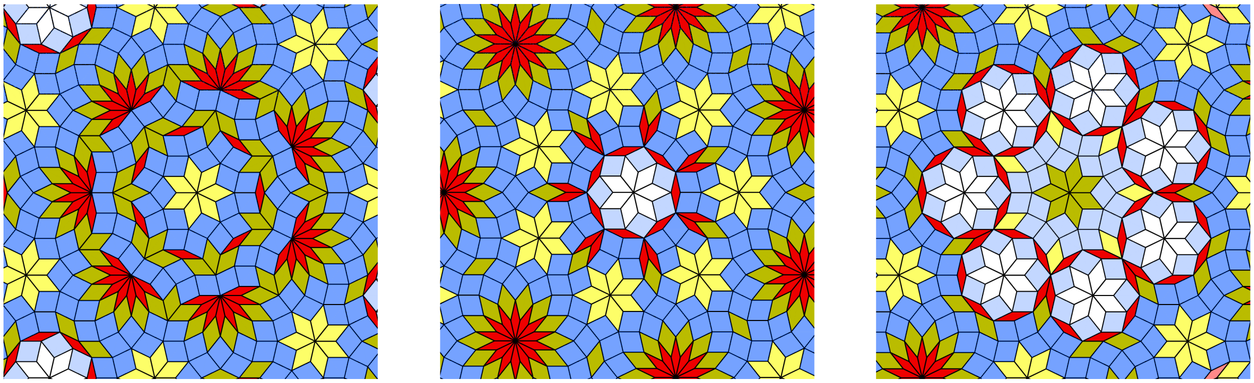

Figure 5: *The other three centers of seven-fold rotational symmetry. The center of each pattern is a rosette of seven B rhombi, see Figure 3. Inflation exchanges them cyclically from left to right. Images generated by a public browser app [8].*

Summary and Conclusions

We examine the substitution method for creating seven-fold quasiperiodic tilings with three different rhombi as tiles. Applying Ptolemy's theorem to a regular heptagon we obtain that the product of its two diagonals and also squares of diagonals are sums of diagonals and the side of the heptagon. Thus, linear combinations with integer coefficients of the side and the two diagonals of a regular heptagon make up a multiplicative ring H. The ratios between the sides and one diagonal of the rhombus tiles are also elements of H. From this follows that the inflation ratios r of most seven-fold tilings are elements of H because r is directly related to the arrangement of small rhombi at the border of inflated tiles. A given inflation ratio determines the numbers of the various small rhombi replacing an inflated rhombus tile and the substitutions at its diagonals and borders. We present a tiling and show how these results are used. Our approach can be used to create tilings with other rotational symmetry of odd order, such as nine-fold rotational symmetry.

References

[1] L. Danzer. https://tilings.math.uni-bielefeld.de/substitution/danzers-7-fold/.

[2] C. Goodman-Strauss. https://tilings.math.uni-bielefeld.de/substitution/goodman-strauss-7-fold-rhomb/.

[3] A. E. Madison. https://tilings.math.uni-bielefeld.de/substitution/madison-7-fold/.

[4] D. S. Richeson, "Tales of Impossibility", pp. 137–143, Princeton University Press, Princeton, 2019.

[5] T. P. Schaad. "A Minimal 7-Fold Rhombic Tiling." https://arxiv.org/abs/2006.03453.

[6] T. P. Schaad. "A Challenging 7-Fold Puzzle." https://arxiv.org/abs/2112.00625.

[7] J. Socolar. https://tilings.math.uni-bielefeld.de/substitution/socolars-7-fold/.

[8] P. Stampfli. http://geometricolor.ch/qpg/quasiperiodicGenerator/quasiperiodicAndFractal.html.

[9] P. Stampfli and T. P. Schaad. "Quasiperiodic Tilings with 12-Fold Rotational Symmetry Made of Squares, Equilateral Triangles, and Rhombi." *Bridges Conference Proceedings*, Phoenix, USA, 2021, pp. 315–318. http://archive.bridgesmathart.org/2021/bridges2021-315.html.

[10] P. Steinbach. "Sections Beyond Golden." *Bridges Conference Proceedings*, Winfield, USA, 2000, pp. 35–42. https://archive.bridgesmathart.org/2000/bridges2000-35.pdf.

[11] G. J. Toomer. "Ptolemy's Almagest, Translated and Annotated." Gerald Duckworth, London, 1984, p. 50. https://archive.org/details/PtolemysAlmagestPtolemyClaudiusToomerG.5114_20181.