

Single Line Apollonian Gaskets for Fashion

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Abstract

We describe the design of single-line approximations and fractals based on the Apollonian gasket, which is the limit case of configurations of kissing circles. Rather than plotting the circles as discs on a differently colored background (the traditional representation), we explore strategies of drawing all circles as one line without lifting the pen and without crossing itself. We find various drawing strategies, having different aesthetic flavors. We apply fashion tech to design a garment, exploring the implications and potential of the new forms for fashion.

Introduction

In *The Fractal Geometry of Nature* [6], Benoît Mandelbrot writes that the Sierpiński gasket can be knitted with a single loop of thread and that the same is true of the Apollonian net. Mandelbrot presents neither full details of the loop, nor needlework. In *Indra's Pearls* [7], Mumford, Series and Wright generate fractal loops by iterating Möbius transforms. Single-line art fits fashion-tech machines: laser cutters, embroidery machines etc. In this paper, the authors make the “single loop” visible and play with its implementations.

An Apollonian gasket arises from a configuration of kissing circles. We start with one outer circle and two arbitrary mutually touching circles inside. Each triple of mutually tangent circles gives rise to two new circles (e.g. 0, 1 and 2 give rise to 3 and 4). The process can be repeated, gradually adding more and more circles in the gaps between the already constructed circles. Imagining the circles being punched out from a piece of sheet material, the result resembles the typical head gaskets deployed in car and motorcycle engine production and repair; see Fig. 1 (a). It is practical to stop the production process by imposing a minimum radius. The process can go on forever, and then it gives rise to a fractal known as the Apollonian gasket.

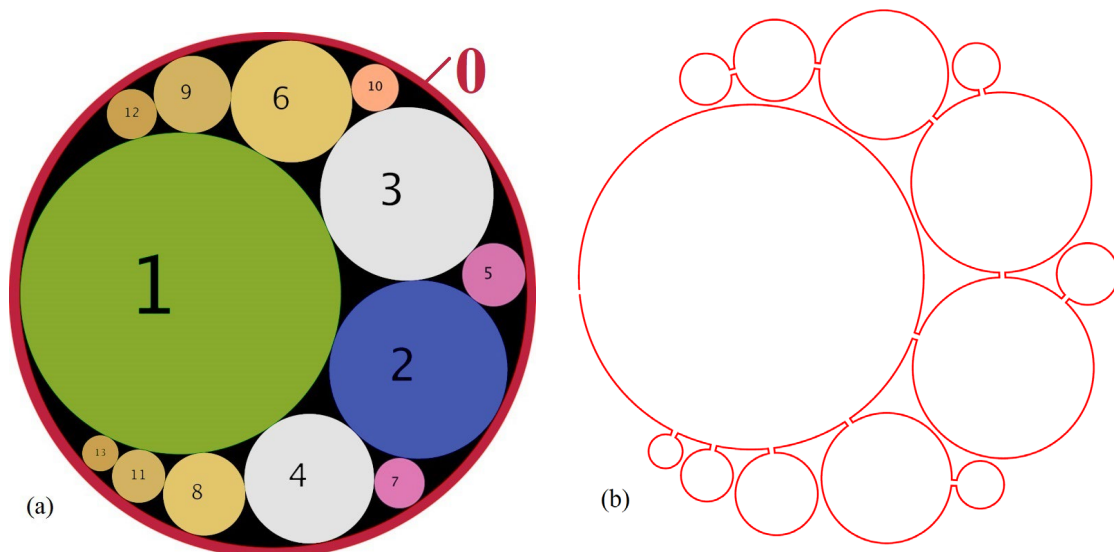


Figure 1: Circles in Apollonian configuration: (a) the outer circle has number 0, then 1 and 2 are placed manually; the others follow as 3 mutually tangent circles define 1 or 2 others, (b) line tracing.

Many fractals have a tree-shaped hierarchy (e.g. Pythagorean tree, Koch curve, Sierpiński triangle), but the Apollonian hierarchy is more complex. An inner circle has *three* parents, e.g. 0, 2, and 4 are parents of 7 in Fig. 1 (a), where we use additive color-mixing to highlight the parent-of-relationship. Fig. 2 shows the graph of this relationship, plotted with Mathematica. The rich hierarchy offers freedom when designing trace lines.

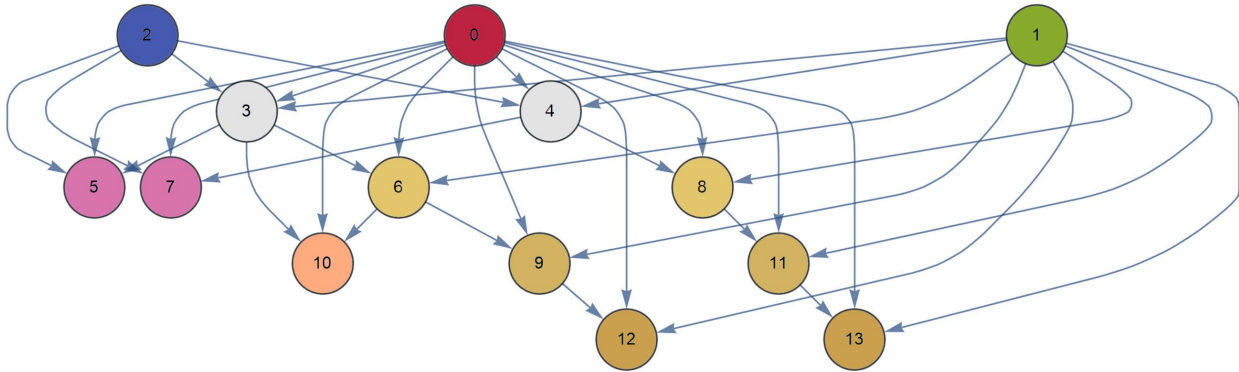


Figure 2: Graph of the parent-of relation of the circles from Figure 1.

We find the inner circle c_4 from three mutually tangent circles c_1 , c_2 , and c_3 as follows. Its radius $r_4 = (k_4)^{-1}$ is given by the elegant equation

$$k_1^2 + k_2^2 + k_3^2 + k_4^2 = \frac{1}{2}(k_1 + k_2 + k_3 + k_4)^2$$

which is known as Descartes' theorem, rediscovered by Frederick Soddy in 1936 [8]. Each k_i is the curvature, i.e., the reciprocal of the radius r_i . Once we have r_4 , the coordinates of c_4 are easily found. An alternative approach is given in [4]: a tiny circle is positioned in each gap and then iteratively grown and maneuvered until it is tangent to its parents. In the present project we use the exact method of Descartes and Soddy.

Towards a Line Representation of the Fractal

Our approach of presenting an Apollonian gasket as a line fractal is as follows. First, generate a set of circles by building inner circles down to a certain minimal radius, until all possibilities for adding more circles are exhausted. Next, construct for each circle the list of circles touching it, sorted by angular position, going counter-clockwise. We call the circles in this list the *satellites*. We choose an entry point for circle 1, say at -179° (almost nine o'clock). Then going round along circle 1 we trace its contour, going counter-clockwise, and enter a satellite when passing by at the point of tangency, and recursively trace that satellite too before continuing to trace the rest of circle 1 (and more satellites).

For generating the circles, we re-used parts of *Interactive Apollonian* by David Crooks [2]. Sorting the satellite list and the recursive step are new, coded in Processing 3. Before plotting, we shrink the circles by a small distance (reducing the radius by δ) and give a width (2δ) to the entry, so it appears as a bridge connecting circle and satellite. We are free in the strategy choosing which satellites to trace immediately, and which to skip. A circle appears in at least two satellite lists, some even more. The outermost circle is not traced. For the construction of an Apollonian configuration we use a radius-based *stop-criterion*, which means that we choose a number R_{min} so that all possible circles with radius $\geq R_{min}$ are included, and no circles having radius less than R_{min} . The effect is a certain uniformity in visual appearance, as can be seen in Fig.1 (b), Fig. 3 (a) and Fig. 4 (a).

Two examples of strategies are in Fig. 3 and Fig. 4. The former is hierarchical: break out to *direct children* only (don't skip generations). The latter strategy is greedier: select *any* satellite whose radius is not more than 1.2 times the radius of the circle from which it branches.

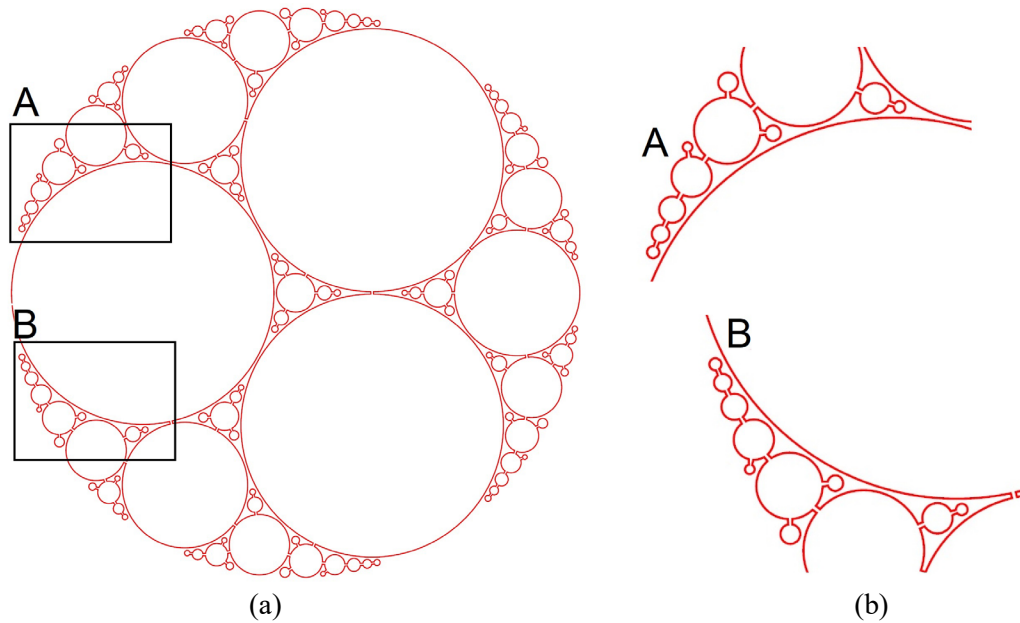


Figure 3: Hierarchical strategy of selecting satellites for recursive tracing: (a) all circles, (b) blow-up of selected regions.

We prefer the second strategy, as shown in Fig. 4 for aesthetic reasons: there is more variation in the molecule-like configurations and this variation breaks the overall symmetry in a playful manner.

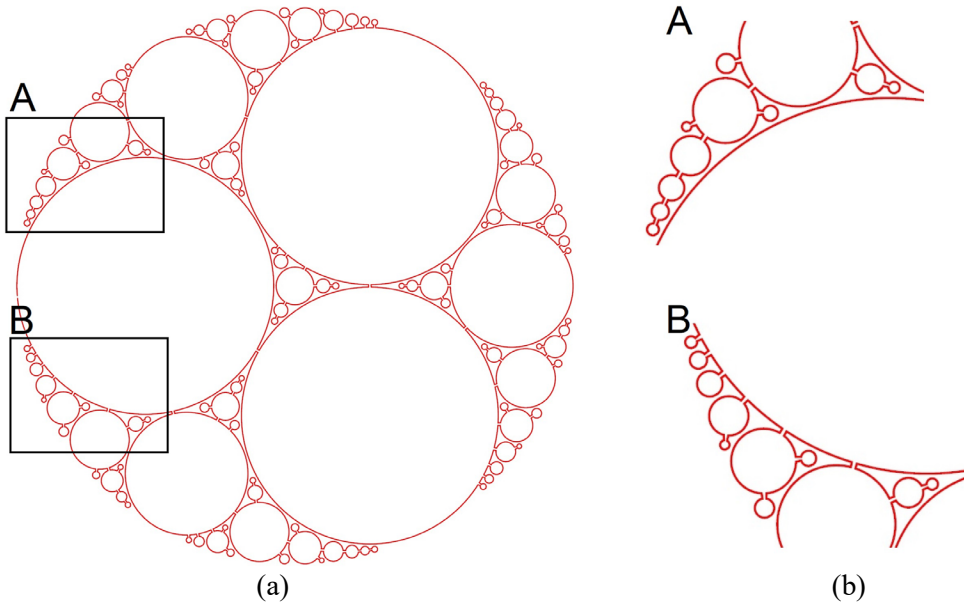


Figure 4: Greedy strategy of selecting satellites for recursive tracing: (a) all circles, (b) blow-up of selected regions.

The object-oriented code for class Circle is organized such that the strategy is easily adjusted, as shown in Fig. 5. The `gen` field of a circle denotes its *generation*, assigned during the initial construction phase.¹ Note the provision to prevent reentrant tracing. We can also play by limiting the recursion depth (the recursive procedure has a “budget” parameter, as usual). Both the hierarchical and the greedy strategy trace every circle. Budgeting strategies may leave some circles unreachable, which is easily found by testing.

```

boolean eligible(Circle sat){
    return !sat.entered
        && this.gen <= sat.gen
        && sat.gen - this.gen <= 1;
}
(a)

```

```

boolean eligible(Circle sat){
    return !sat.entered
        && sat.r < 1.2*this.r;
}
(b)

```

Figure 5: Satellite selection code defining the strategies: (a) hierarchical strategy, (b) greedy strategy.

Filling the Gaps

For many artistic and practical applications it is an unattractive feature of the Apollonian gasket that it has large areas where nothing happens (the three main circles). Already in medieval Gothic architecture this limitation was noted and small circles were placed inside the large ones, as discussed in the paper by Tiffany Inglis and Craig Kaplan [5]. Francesco De Comit  [1] went further, designing aesthetically pleasing fractals, “avatars of the Apollonian gasket”. Also for our own fashion-tech projects, large openings are sometimes unattractive.

Can we turn the line-based version into a space-filling fractal? We apply recursive circle packing to the interiors of the existing circles as in [1], next apply line tracing again. Here we restrict ourselves to circle configurations in which the three initial circles have the same size. We embed the recursive tracing of the interiors into the overall tracing of the circles and the satellites. We had to overcome two hurdles:

- When a circle is large enough to accommodate an interior, how should one choose the angular position at the circle’s circumference for turning inward? This position should not be close to an eligible satellite nor close to the entry (where drawing the circumference began). The adopted solution is to we look for a suitable inward-turning point by first trying the best options, -60° and 180° (0° means 3 o’clock). Then the nearest of the main interior circles is entered. If these options are unavailable, divert to positions near -60° , for example -63° (allowing the bridge to become slanted).
- Tracing circles in the interior goes clockwise if the circle itself goes counter-clockwise (default), otherwise the turning-inward bridge would become an \times . Rather than coding a clockwise version of the tracer, we keep it *as is*, yet locally flip the coordinate system when turning-inward. We use `beginShape`, `endShape`, and `vertex`, like the operations built in to Processing to make arcs, but wrote new transforms with `pushMatrix`, `popMatrix` etc. as the built in transforms do not work inside shapes. Rotation, translation and scaling are combined in 3×3 matrixes of homogeneous coordinates.

The resulting line drawings are shown in Fig. 6. We see lollipops and triplet fetuses connected by a kind of umbilical cord. The overall appearance is playful. Whereas all inner circles have the same D_3 symmetry, the tracing line is less regular, it breaks the symmetry.

We conjecture that the paths define a curve (where a *curve* is the image of a continuous function $f: [0,1] \rightarrow \mathbb{R} \times \mathbb{R}$ using the Euclidean metric). Eventually, the successive paths will come arbitrarily close

¹ Generation 0 consists of circles 0, 1 and 2 (in Figs. 1(a) and 2). Then generation 1 contains circles 3 and 4. Gen 2 is 5, 6, 7, and 8. Gen 3 is 9, 10, and 11. Gen 4 is 12 and 13. In general, a circle’s `gen` is one added to the highest parent `gen`.

to any point inside the outermost circle. The limit curve has fractal nature. Moreover we conjecture it is space-filling. The details of the explorations into space-filling fractals are too technical for the present paper (and the aesthetics of the curves do not depend on it), so we made the explorations available on arXiv [3].

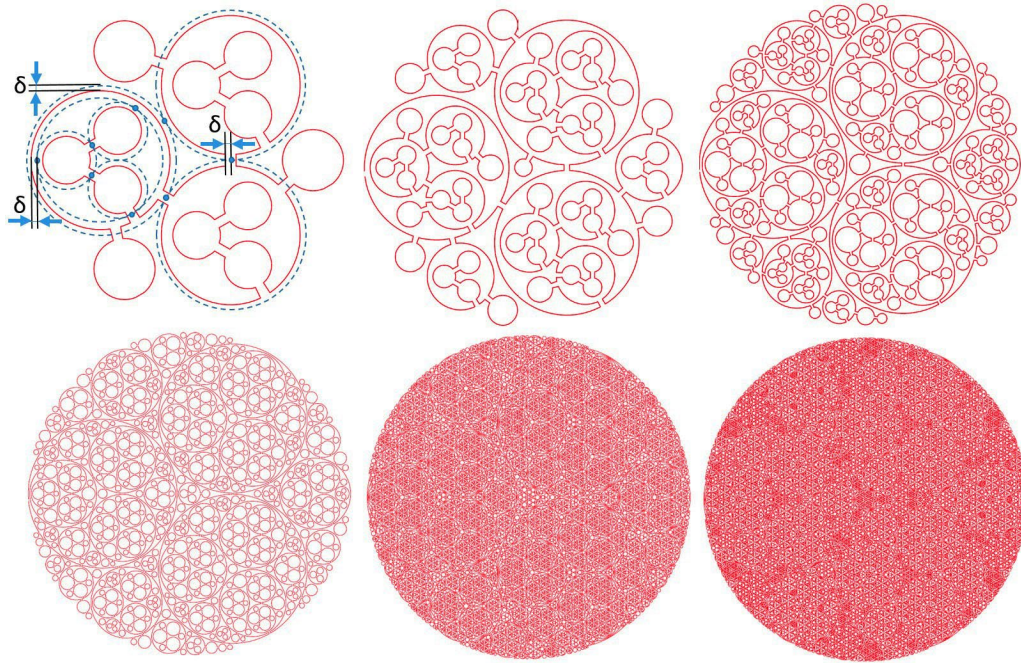


Figure 6: *Apollonian paths: successive versions of the line representation of the Apollonian gasket.*

A single line across an area partitions it into two sub-areas. These areas then are filled by two contrasting colors in Fig. 7. The overall appearance is mesmerizing as the foreground-background Gestalt is unstable.

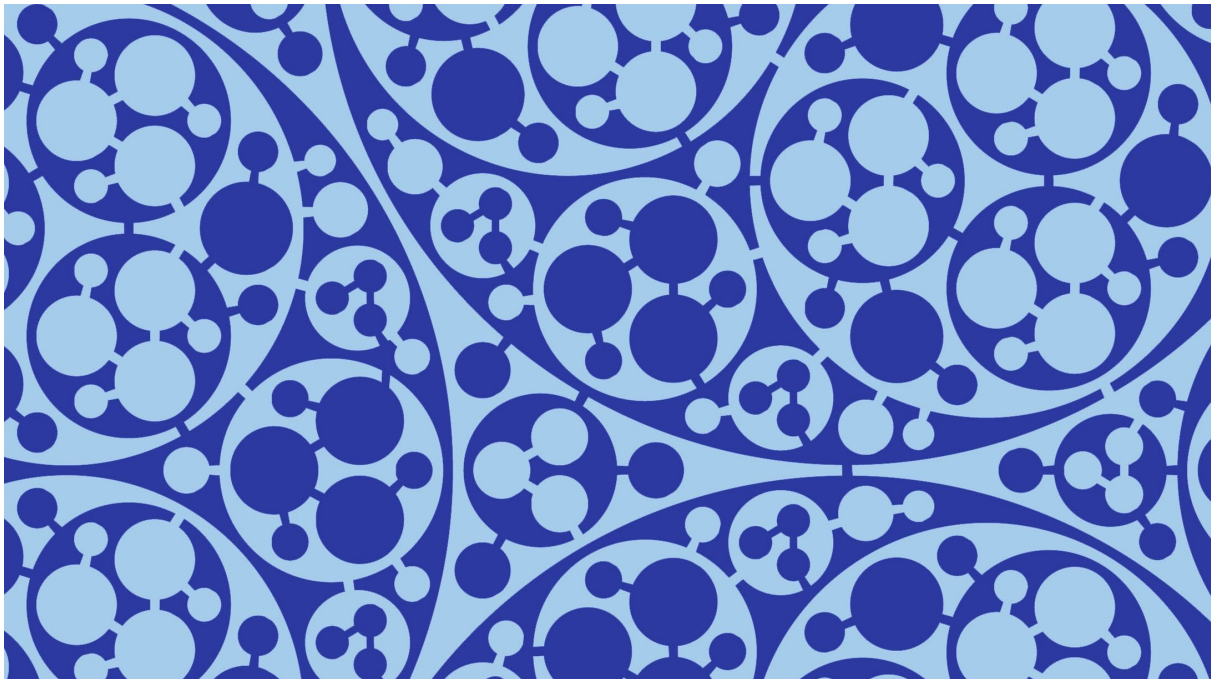


Figure 7: *Fragment of a contrast-colors filled Apollonian path.*

Towards Fashion

From here, we could go in many directions: to make a garment, or exploit the technical options of the long folded line. For the technical options we mention machine-embroidering conductive yarn. We tested this in the TU/e Wearable Senses lab, putting the conductive yarn in the under bobbin. Potential applications include heating pads, capacitive sensors, and fractal antennas. We leave these as options for future research.

For now, we choose to make a garment, a most simple dress. The Apollonian lines provide a motif to be mapped onto the fabric. We use both laser engraving *and* sublimation printing, like an off-register woodblock print. The cut pattern design is informed by the circles in the motif. Fig. 8 shows what “informed by” means. In Illustrator we dive into a (generated) *sea of circles*, looking for a configuration which makes sense. Near the center of the Apollonian line there is a configuration resembling a traditional sleeve head; see Fig. 8 (a). We adapt the sleeve accordingly, make it wider, as shown in Fig. 8, (b), and next adjust the corresponding armhole opening cut of the body. We add seam allowance, remove circle segments outside the sleeve, remove loose ends and neatly close bridges whose circles were removed. We show the design as it goes to the printer in Fig. 8 (c). We have chosen soft complementary colors for the two-tone pattern (a green-red contrast could hurt the eye). The design principle to let the cut pattern be informed by the features of the math motif was explored earlier in the design of our Taxicab Mandelbrot pieces [9].

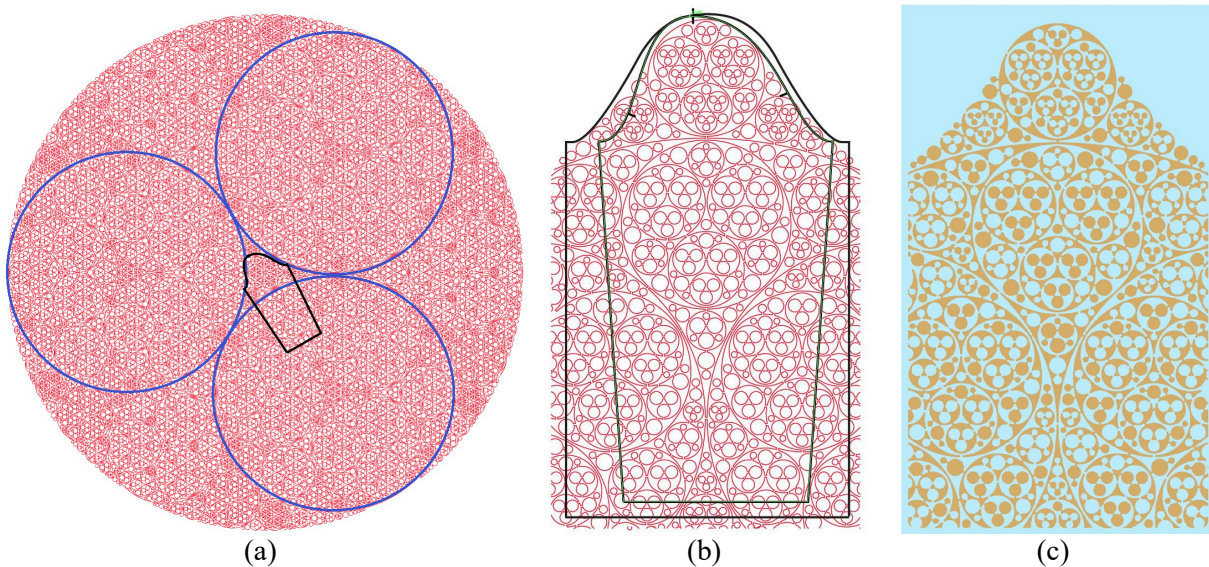


Figure 8: Design of a sleeve cut pattern responding to the form language offered by the Apollonian line: (a) sleeve found in a sea of circles, (b) adapted sleeve, (c) final sleeve design.

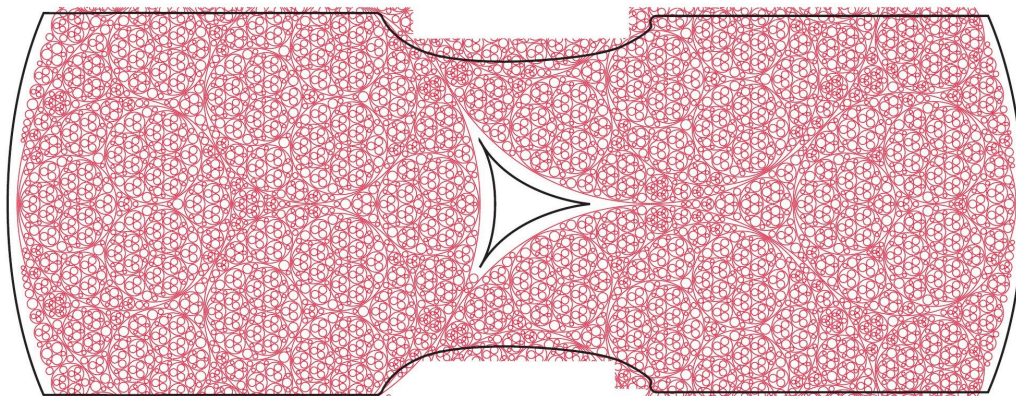


Figure 9: Design of a the neck cut pattern responding to the form language of the Apollonian line.

Another math feature based cut pattern is the neck-line of Fig. 9. Although the first idea was to use a circle again, we found that a triangular area works as well, creating a special collar. In this arrangement, parts of the outer line of the entire fractal match the lower cut lines of the garment; see Fig. 10. We realized a collection of four garments: a short gilet, a longline gilet, circle skirt, and a long sleeve shirt; see Fig. 11.



Figure 10: *The line-Apollonian dress. Model Julia Van Zilt, photographer Beam Contrechoc.*

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Figure 11: *Collection of line-Apollonian garments, photographer: Janne Beelen.*

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