Exploring the Wurzelschnecke: Learning Geometry, Number and Design with the Spiral of Theodorus

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Abstract

In this hands-on virtual workshop, the authors explore some of the educational and design affordances of the (Quadrat) Wurzelschnecke or "square root spiral", also known as the (discrete) Spiral of Theodorus, considering possibilities at different scales and in different media. The workshop will give participants the chance to learn about the geometry and number theory embodied in the Wurzelschnecke by making and re-forming it in a variety of ways. The authors will show artistic and educational experiments they have undertaken with Wurzelschnecke furniture, millinery, fashion design, jewelry and cuisine, and invite participants to help design interactive large-scale museum displays, playground sculptures and movement-oriented activities. Authors will introduce open questions about an analysis of the beautiful 'reverse Wurzelschnecke' spiral and about the optimization of laying out a double Wurzelschnecke on a rectangle of material, and offer an original visual proof of the irrationality of the roots of non-square numbers using an experimental mathematics approach.

Introducing an Ancient Spiral with Educational and Design Potential

The Wurzelschnecke, or Spiral of Theodorus, is a spiral made of discrete right triangles, starting with an isosceles right triangle and then building successive right triangles on the hypotenuse of the previous triangle, while holding the base of each triangle constant (Figures 1 and 2). This spiral dates back at least to Theodorus of Cyrene (465-399 BCE), and is hailed as the first geometrically-constructed spiral [3,4]. Theodorus was the tutor of Plato and Theaetetus in Athens, and a member of the Society of Pythagoras, and was certainly familiar with the theorem we know as the Pythagorean Theorem (known centuries earlier in China as the Gou-Gu ('hypotenuse-leg') Theorem, and familiar in Babylonian mathematics as well).

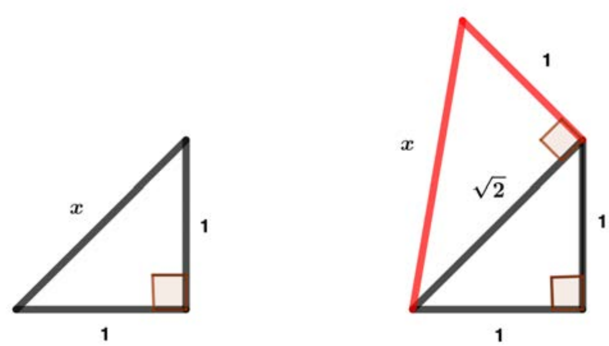

Figure 1: *Steps 1 & 2 of 17 iterations to construct the Wurzelschnecke with straightedge and right angle. Image credit:V. Jungic*

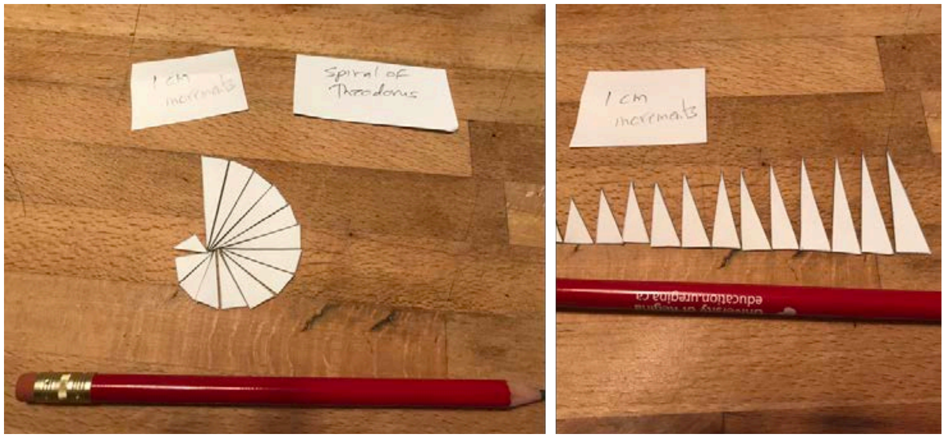

Figure 2: *Getting started with constructing a very small Wurzelschnecke. Photos: S. Gerofsky, 2018.*

The Spiral of Theodorus is named the Wurzelschnecke (literally, ‘root snail’ or ‘square root spiral’) in German. Gautschi [6] differentiates the *Discrete* Spiral of Theodorus (the one we are interested in) from the late 20th century invention of the smooth *Analytic* Spiral of Theodorus (which we will not be considering here.) The more specific German term ‘Quadratwurzelschnecke’ is the German language equivalent of the Discrete Spiral of Theodorus. We will use the terms Spiral of Theodorus and Wurzelschnecke throughout this paper to refer to the discrete and more ancient version of the spiral.

An interesting quote from Plato's book *Theaetetus* refers to Theodorus teaching his two pupils, Plato and Theaetetus, about the incommensurability of the square roots of non-square whole numbers. Plato remarks: “Theodorus was proving to us a certain thing about square roots, I mean the side (i.e. root) of a square of three square units and of five square units, that these roots are not commensurable in length with the unit length, and he went on in this way, taking all the separate cases up to the root of seventeen square units, at which point, for some reason, he stopped.” [8]. There is much speculation about why Theodorus would go up to 17 and then halt. With our interest and experimentation with the Spiral of Theodorus, we throw our support behind those who have noted that this spiral constructs right triangles with hypotenuses of the length of the roots of the successive counting numbers up to 17, after which the spiral begins to overlap itself. There is no reason why the spiral could not be treated as a way to construct roots for *all* the counting numbers, as high as you like, but apparently for Theodorus (or any geometer who is physically drawing and constructing these triangles), seventeen triangles creates one round of the spiral and is a natural and convenient place to stop (Figure 3).

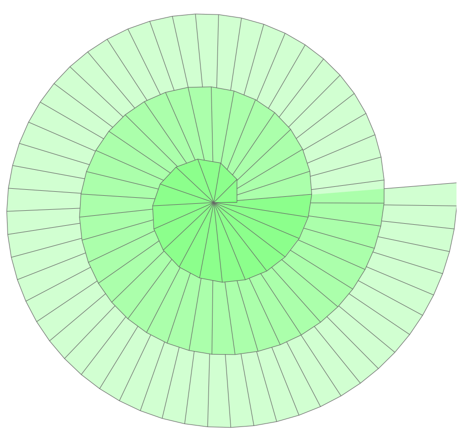

Figure 3: *Multiple iterations of the Wurzelschnecke. The darker green spiral in the centre includes the first 17 triangles known to Theodorus. Image credit: Extended spiral of Theodorus by Pbroks13 - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index*

The Wurzelschnecke, which Gautschi calls a “harmonious, very pleasing and elegant spiral” [6] has captured the interest and imagination of one of our co-authors, Gerofsky, since it appeared in a Bridges Mathematical Art Galleries sculpture by Bernard Rietzl in 2016 [9]. Gerofsky has her preservice secondary math teacher students do a math art project every September, based on that summer's Bridges Gallery works, with a participatory element for the rest of the class. A project group in the 2016 class worked with Rietzl and his *Nautilus Theodori*, teaching the class how to construct the simple and elegant Wurzelschnecke spiral as a pencil drawing. Other Bridges authors have mentioned the Spiral of Theodorus in previous papers, but have not explored this figure in depth [7, 10]. Gerofsky has had a growing fascination with the Wurzelschnecke, and has experimented with it over the past three years, at different scales (from miniature to furniture-sized) and in a variety of media (paper and corrugated cardboard, uncurling basswood triangles on a hat, small and large foam and fabric, upcycled metal biscuit tins, and edible pastry Wurzelschnecke) (see Figures 4, 5 and 6).

Figure 4: *Wurzelschnecke furniture, sculpture, architecture. Photos: S. Gerofsky, 2019*

Figure 5: *Prototyping Wurzelschnecke jewelry with biscuit tins. Successive right triangle ‘pennant’, Wurzelschnecke, reverse Wurzelschnecke. Photos: S. Gerofsky, 2019.*

She has led workshops using these iterations of the Wurzelschnecke, exploring its possibilities for number theory and design, with groups of high school and university students and mathematics educators. Photos and short video clips from these workshops and Wurzelschnecke models and artwork will be shared with our Bridges participants. Participants will have the chance to experiment with the affordances of the Wurzelschnecke as a design element in different configurations, including patterns reminiscent of the Towers of Theodorus [11] and spiral-inspired designs in [12]. Robertson and Jungic were both involved in a recent participatory workshop led by Gerofsky at the Banff International Research Station's Geometry: Education, Art, Research online symposium (BIRS GEARS), and were inspired to create this expanded workshop for Bridges 2021, incorporating aspects of mathematical proof and inquiry, mathematics education at the secondary level, and geometry-based design inquiry around the Wurzelschnecke.

Figure 6: *Edible pastry Wurzelschnecke and cardboard models.Photos: S. Gerofsky, 2021.*

Co-author Jungic is interested in using experimental and constructivist proofs pioneered by his Simon Fraser University colleague Jonathan Borwein [1, 2] as approaches to proofs for learners and for mathematicians interested in constructivist and experimental methods. He was inspired by the construction of the Wurzelschnecke to create a simple and elegant new visual proof using *reductio ad absurdum* of the irrationality of the square root of any non-square positive integer, using the Spiral of Theodorus construction as its basis. Jungic's proof in the general case is included below. Interestingly, Jungic has found a similar approach (using diminishing similar triangles, although not invoking the Spiral of Theodorus), in a speculative comment from Heath [5] on Theodorus. Here is a key section of Jungic's visual proof by Wurzelschnecke:

In the proof below we will use the following notation and terminology. For the given points O and X_1 and $n \in \mathbb{N}$, by $TS(O, X_1; n)$ we will denote the finite Spiral of Theodorus $O, X_1, \dots, X_n$ such that if $|\overline{OX_1}| = x$ then $|\overline{X_i X_{i+1}}| = x$ with $\overline{OX_i} \perp \overline{X_i X_{i+1}}$, for $i \in \{1,2,\dots,n-1\}$. We will say that $\overline{OX_1}$ is the initial segment and that $\overline{OX_n}$ is the terminal segment of the spiral $TS(O, X_1; n)$.

We observe that by using the Pythagorean Theorem $n-1$ times, we obtain that $|\overline{OX_n}| = x\sqrt{n}$. This is the reason that the Spiral of Theodorus is sometimes called the 'square root spiral.'

Suppose that $n \in \mathbb{N}$ is not a perfect square and that there are positive integers p and q, $q < p$, such that $\sqrt{n} = \frac{p}{q}$.

We choose point O and A_1 so that $|\overline{OA_1}| = q$ and construct the Theodorus Spiral $TS(O, A_1; n)$. Then

$$|\overline{OA_n}| = p = \sqrt{n} \cdot |\overline{OA_1}| = q\sqrt{n}.$$

Let $q_1 = p - \lfloor\sqrt{n}\rfloor \cdot q$, where $\lfloor\sqrt{n}\rfloor$ denotes the largest integer smaller than $\sqrt{n}$. Since $p = q\sqrt{n}$ it follows that

$$q_1 = p - \lfloor\sqrt{n}\rfloor \cdot q = q\sqrt{n} - \lfloor\sqrt{n}\rfloor \cdot q = \left(\sqrt{n} - \lfloor\sqrt{n}\rfloor\right) \cdot q.$$

Since n is not a complete square, we have that $0 < \sqrt{n} - \lfloor\sqrt{n}\rfloor < 1$. This establishes that $0 < q_1 < q$ and $q_1 \in \mathbb{N}$.

We denote by B_1 the point on the line segment OA_1 such that $|\overline{OB_1}| = q_1$. Let $p_1 = |\overline{OB_n}|$ be the length of the terminal segment of the Theodorus Spiral $TS(O, OB_1; n)$. As before, $p_1 = q_1\sqrt{n}$.

Next, we establish that $p_1$is a positive integer less than p. From

$$0 < p_1 = \sqrt{n} \cdot q_1 = \sqrt{n} \cdot \left(p - \lfloor\sqrt{n}\rfloor \cdot q\right) = \sqrt{n} \cdot \left(\sqrt{n}q\right) - \lfloor\sqrt{n}\rfloor \cdot p = nq - \lfloor\sqrt{n}\rfloor \cdot p$$

it follows that $p_1 \in \mathbb{N}$. Finally, from

$$p_1 = nq - \lfloor\sqrt{n}\rfloor \cdot p = \sqrt{n} \cdot p - \lfloor\sqrt{n}\rfloor \cdot p = (\sqrt{n} - \lfloor\sqrt{n}\rfloor) \cdot p$$

we conclude that $p_1 < p$. (See Figure 7.)

Hence if we assume that $\sqrt{n}$ is irrational then we can construct a diminishing sequence of spirals $TS(O, X_1; n)$ with the property that each of them is with the initial and the terminal segments of lengths being integers with the ratio of $\sqrt{n}$. In particular, this construction establishes the sequence

$$|\overline{OA_1}| = q > |\overline{OB_1}| = q_1 > |\overline{OC_1}| = q_2 > \ \ldots$$

This contradicts the fact that any decreasing sequence of positive integers must terminate after a finite number of steps. Hence, if n is not a perfect square than $\sqrt{n}$ is an irrational number.

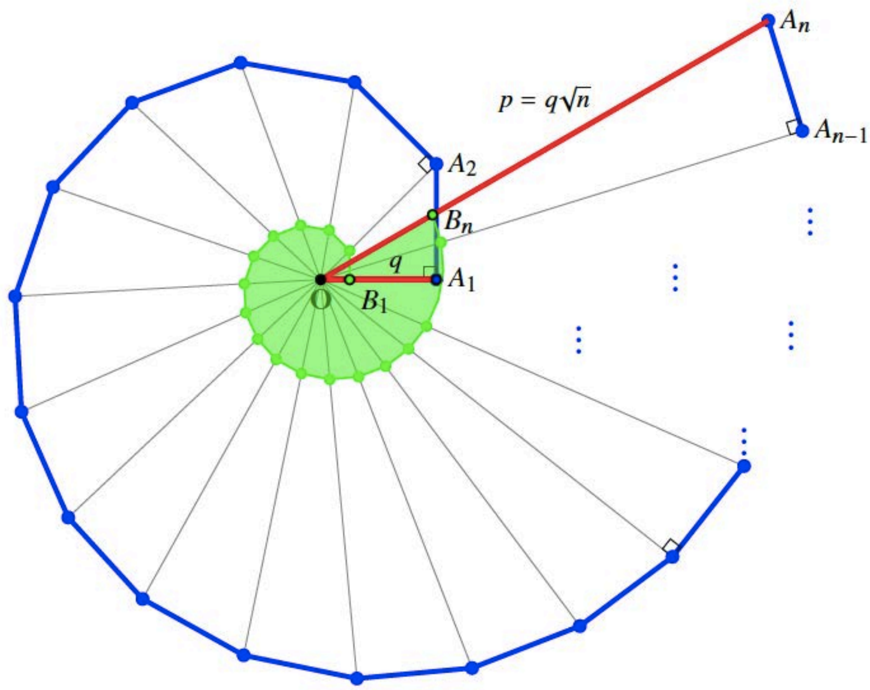

Figure 7: *Visual proof (by diminishing Wurzelschnecke) of irrational roots of non-square natural numbers. Image credit: V. Jungic*

Co-author Robertson has been collaborating with Gerofsky over the past year in brainstorming embodied, large-scale mathematical manipulatives that could be used in museums, classrooms, playgrounds, parks and festivals to engage members of the public with intriguing movement-oriented activities that inspire a deeper understanding of mathematical patterning. Robertson and Gerofsky will share models and plans for some of these large-scale, manipulable sculptural pieces based on the Wurzelschnecke, along with prototypes for

Wurzelschnecke jewelry, clothing, a 3D printed cookie cutter and other applied designs that allow for geometric and number theory explorations (Figures 8 and 9).

Figure 8: *Prototype for Fedoras of Theodorus (with Marcia Martinez): kinetic geometric millinery. Photos: S. Gerofsky, 2018.*

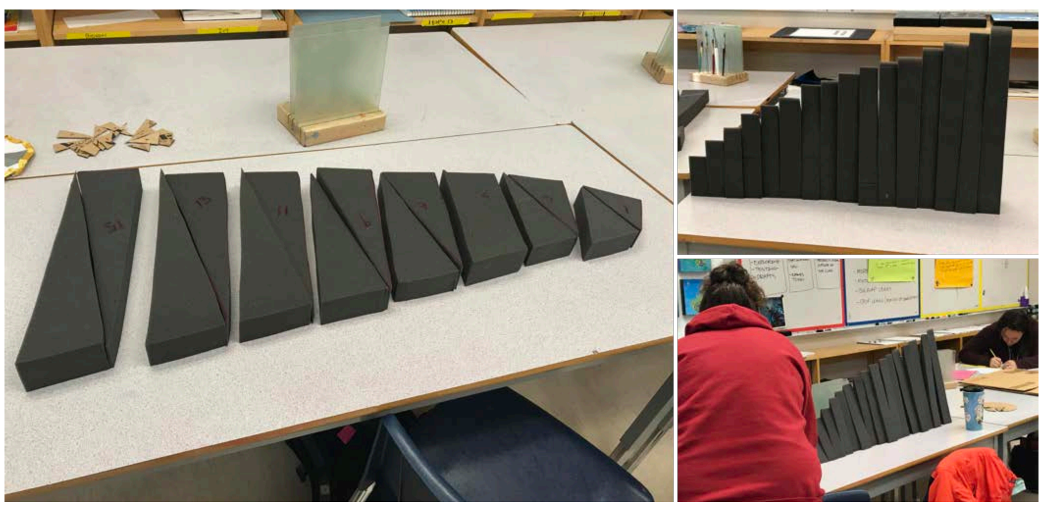

Figure 9: *Students and teachers exploring the linear profile of Wurzelschnecke triangles as a square root function; Tower of Theodorus. Photos: S. Gerofsky, 2019.*

The co-authors have an interest in open questions about an as-yet-unexplored geometric shape, the reverse Wurzelschnecke, formed by placing the Wurzelschnecke right triangles in reverse order (with the hypotenuse of the subsequent triangle set on the leg of the previous one, rather than on its hypotenuse). This geometric shape, which Gerofsky encountered by chance through experimentation in participatory workshops, encloses an almost-circular hole that is aesthetically very pleasing. But why does a series of (very square) right triangles construct such a smooth, pleasing circle-like shape? We plan to lead workshop participants in an exploration of this open geometric question.

Planned Workshop Activities

- Introduction to the Wurzelschnecke/ Spiral of Theodorus and its history.
- Participants to construct their own Wurzelschnecke from paper or cardboard, ruler and right angle, and scissors. These Wurzelschnecke models will be used to explore the construction of the roots of the natural numbers and other theoretical and design possibilities with this geometric shape. We will also explore the open question of how to optimize the layout of a pair of Wurzelschneckes on a rectagular surface (Figure 10).
- Introduction to works we have created incorporating the Wurzelschnecke, reverse Wurzelschnecke, Towers of Theodorus, Fedoras of Theodorus, Wurzelschnecke pastry, fashion and jewelry. Discussion with participants about ways to use these resources for teaching and learning mathematics and geometry-based design (Figure 11).
- Exploration of the reverse Wurzelschnecke and working towards a rationale/ proof for its intriguing shape, using participants' paper/ cardboard models for experimentation.
- Exploration of a new visual proof of the irrationality of the non-square natural numbers using the Wurzelschnecke construction.
- Brainstorming further large-scale possibilities for public sculpture, playground features and manifestations of the Wurzelschnecke in other media to promote public mathematical appreciation and understanding.

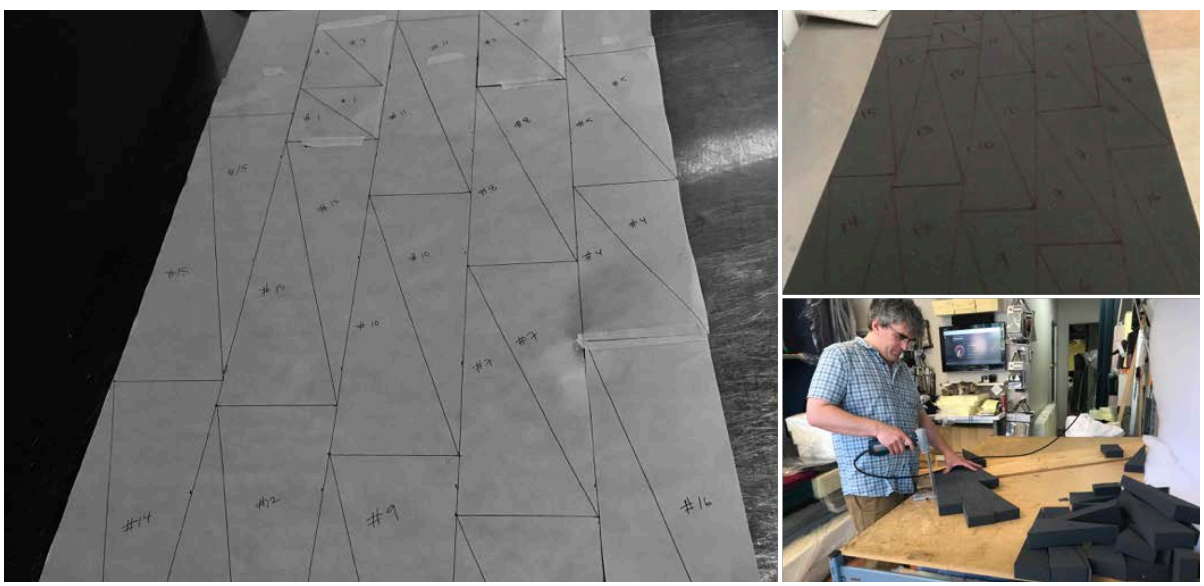

Figure 10: *Open question: how to optimize the layout of the (two) Wurzelschneckes on a rectangular surface? Photos: S. Gerofsky, 2018.*

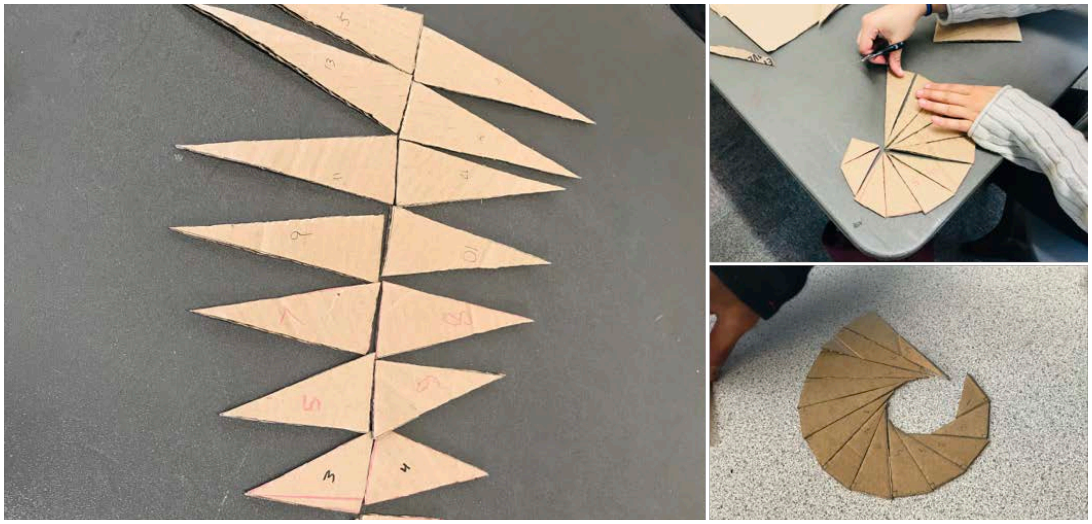

Figure 11: *Secondary students exploring (clockwise from left) inverted Tower of Theodorus, Wurzelschnecke, and reverse Wurzelschnecke. Photos: S. Gerofsky, 2018.*

Summary and Conclusions

The Wurzelschnecke or Spiral of Theodorus is a simple geometric construction with great potential for educational applications, in terms of the history of mathematics, the development of the concept of irrational/ incommensurable numbers through geometric construction, and the interplay between interesting geometries and all kinds of design. We hope that this hands-on workshop, that brings together practical, theoretical and exploratory/ speculative aspects of mathematical design, will inspire everyone involved to design learning opportunities and beautiful, interesting artwork of all kinds with the Wurzelschnecke.

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