Space-Filling, Self-Similar Curves of Regular Pentagons, Heptagons and Other n-Gons

Stefan Pautze

Visualien der Breitbandkatze, Pörnbach, Germany, stefan@pautze.de

Abstract

We sketch a method to derive space-filling, self-similar and simple (one stroke) curves of regular n-gons from decorations on cyclotomic aperiodic substitution tilings which are also stone inflations. Two examples are introduced, a space-filling, self-similar curve of the regular heptagon and an FASS curve of the regular pentagon.

Introduction

The research of self-similar, space-filling curves was mainly driven by their mathematical applications, in detail their ability to map one-dimensional line segments to two-dimensional areas and vice versa as already noted by G. Peano [16] and D. Hilbert [12]. They have a wide use in information technology [3] and can be classified as a special class of fractals [14]. The author's motivation for the recent research goes beyond that. Self-similar, space-filling curves have great aesthetic and artistic qualities. For an example see Figure 1.

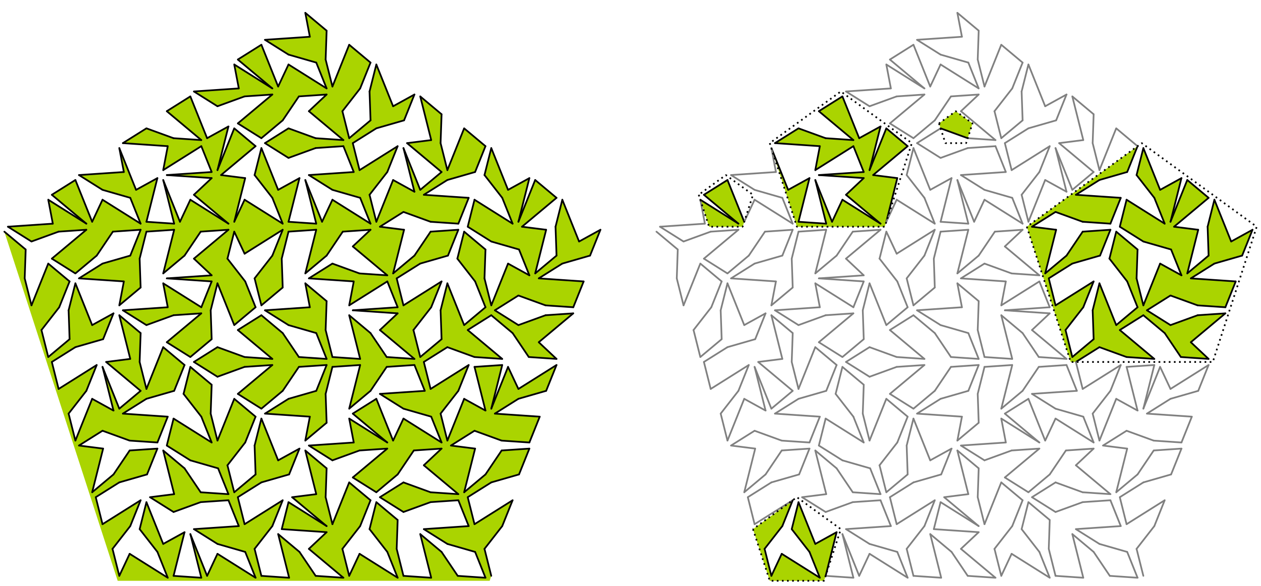

Figure 1: *The 6th iteration of a space-filling, self-avoiding, simple and self-similar (FASS) curve of the regular pentagon is shown on the left side. It divides the pentagon into two parts (green and white) and it contains segments which also cover areas in the shape of pentagons. Some of these segments in different sizes are marked on the right side of the figure.*

It is no surprise that Bridges conferences already received and accepted submissions in this field. J. Arndt and J. Handl [1] as well as J. Ventrella [23] contributed such articles to Bridges Linz 2019. During a discussion at the conference with J. Arndt, J. Handl and J. Ventrella the question was raised whether self-similar, space-filling curves of regular n-gons exist for $n \geqq 5$.

In principle it is easy to find a space-filling curve for any (regular) n-gon by dissecting it into a sequence of triangles, applying the Polya curve [19] to each of them and joining the Polya curves on the triangles together. The challenging part is to achieve a curve which is space filling and self-similar. These and the following mathematical terms are introduced in the next chapter in detail.

One could argue that only regular triangles and squares can be dissected into smaller copies of themselves and that no other regular n-gons have self-similarity on their own. However, some structures exist which yield self-similarity and areas in the shape of regular n-gons. This may be the case in general for aperiodic substitution tilings as described in [2, and references therein] and in detail for cyclotomic aperiodic substitution tilings as discussed in [15, and references therein]. As we will see later, the space-filling, self-similar curves introduced in this article contain infinitely many segments which cover areas in the shape of the regular n-gon at a smaller scale. It is also possible to re-embed the curves into appropriate curves of higher order. Within this paper we will consider this as sufficient to assume self-similarity.

The basic idea of this article is to derive curves with the desired properties from decorations on cyclotomic aperiodic substitution tilings which are also stone inflations.

Some articles have discussed similar approaches before. According to J. Ventrella [22], space-filling, self-similar curves on the Pinwheel tiling were discovered independently by A. Goucher and G. Teachout [21], but the only prototile of the pinwheel tiling has the shape of a right triangle. Decorations were applied on the Ammann A5 tiling as described in [8, Chapter 10.4] and [4] to obtain an FASS curve in [9] by R. Hassel. However, the areas covered by his curve have fractal boundaries and thus are not able to cover n-gons with straight boundaries. Finally F. Henle suggested decorations on the Robinson Tiling [10]. A closer inspection reveals that the pentagonal shape of the curve is enforced by the initiator (the initial setting) and no other segments which cover regular pentagonal areas appear on a lower scale. As a consequence Henle's pentagonal curve is not considered as valid solution for the problem addressed in this article.

This paper will start with some definitions, followed by the sketch of the method and two examples. Finally the article will be concluded with an outlook for the further research.

Terms and Definitions

A space-filling curve is usually defined as a map of a one-dimensional line segment to a continuous curve that covers a two-dimensional area or even n-dimensional space and vice versa. In our case, self-similarity, self-avoidance and simplicity are considered desired properties. Self-similarity means here that the structure of the curve is up to the scale identical to the structure of its parts. A curve is self-avoiding if it does not cross or touch itself. Simplicity means here that the curve can be drawn with one stroke. A space-filling curve which is self-avoiding, simple and self-similar is referred to as an FASS curve.

Let's skip the terms "decoration", "cyclotomic", "aperiodic" and "stone inflation" for now and continue with the substitution tiling which will provide the underlying structure of the curve. A substitution tiling is defined by a set of prototiles which can be expanded with a linear map - the "inflation multiplier" - and dissected into copies of prototiles in original size - the "substitution rule" as shown in Figures 3 and 4. Substitution tilings allow covering the entire Euclidean plane without gaps and overlaps, obviously with tiles of finite size. In our case we will "tile" a finite area with tiles of infinitesimal size. In other words we omit the expansion and apply the substitutions on a single prototile, preferably in the shape of a regular n-gon.

Furthermore, the substitution tiling must be a so called "stone inflation" in the sense of L. Danzer as described in [2, p. 148]. In other words, the result of n substitution cycles applied to a single prototile - a so called level-n super tile - has still the same shape as the initial prototile. If this condition is not met, the boundaries of the level-n supertiles get more and more fuzzy - or more precisely fractal - with increasing n, as in the case of Hassel's curve [9] which we noted already above.

Remark. As a matter of fact some of the most famous aperiodic substitution tilings such as the three variants of the Penrose Tiling i.e. the "Starfish, Boat and Diamond Tiling" [17], "Kite and Dart Tiling" [7] and "Rhomb Tiling" [18] are not stone inflations. Their substitution rules place some of the tiles at the edges and produce overlaps between neighboring tiles during the substitution process. To ensure that a valid substitution tiling is created the substitution rules of neighboring tiles must have matching edges and orientations.

Moreover, the substitution tiling must be cyclotomic. That means all vertices are supported by the n-th cyclotomic field for even n and by the $2n$-th cyclotomic field for odd n. This is a necessary condition to allow substitution tilings to have patches with n-fold cyclic or dihedral symmetry such as tiles in the shapes of regular n-gons.

It seems every space-filling, self-similar curve can be described as a decoration on an aperiodic substitution tiling. This class of tilings can cover the entire Euclidean plane and no translation exists that maps the tiling on itself.

Remark. In this paper we discuss the derivation of a space-filling, self-similar curve from an aperiodic substitution tiling. The opposite way is also possible in many cases, so that curve and tiling form a dual structure. The aperiodicity of the structure seems to be a consequence of the combination of the meandering of the curve and the hierarchic properties of the structure as briefly discussed in [20]. This is also true for the well known space-filling curves found by G. Peano [16], D. Hilbert [12] and B. Gosper [6]. Space-filling curves were mainly discussed for their ability to map one-dimensional line segments to two-dimensional areas and vice versa. Their close relationship to aperiodic substitution tilings was not noted at that time and so it took another 76 years until Berger described the first aperiodic tiling, the Wang Tiling [5].

Finally the decorations have to be applied to each prototile. In detail a line is applied to each prototile which connects two points on the border of the tile. The lines on the prototiles and the substitution rules must be adjusted, so that after a substitution each supertile still yields an uninterrupted curve, or more exactly, an uninterrupted polygonal chain. The author has to admit that this problem is rather difficult to solve and that he is not aware of a generic solution. However, for a trial and error approach the following method turned out to be helpful.

Sketch of the Method by Examples

To begin we choose a cyclotomic aperiodic substitution tiling as described in [11] or [15, Chapter 4]. They are available for many odd $n \geqq 5$. The even n cases may be derived from other examples, such as the aperiodic rhomb tilings described in [13] and [15, Section 5], by halving both prototiles and substitution rules. All the suggested aperiodic substitution tilings have the desired properties as noted above. Furthermore, each prototile has the shape of an isosceles triangle, all triangles have an uniform isosceles length and all its inner angles are multiples of π/n. These properties have great advantages. On the one hand isosceles triangles with uniform isosceles length can be generalized for any n. On the other hand they can easily be combined to form patches in the shape of regular n-gons or in the shape of isosceles triangles in a larger scale.

In the following step by step description we will focus on the case $n = 7$. We start with the aperiodic substitution tiling in the first two lines of Figure 2 taken from [15, Figure 5]. It has three prototiles P_1^*, P_2^* and P_3^*, three corresponding substitution rules and also three corresponding level-2 supertiles.

In the first step we rearrange at least one of the level-2 supertiles so that it contains a patch in the shape of a regular n-gon. This is shown for the level-2 supertile of the prototile P_2^* in the shape of the most obtuse triangle as displayed in the third line of Figure 2. It can be shown that this approach also works for the

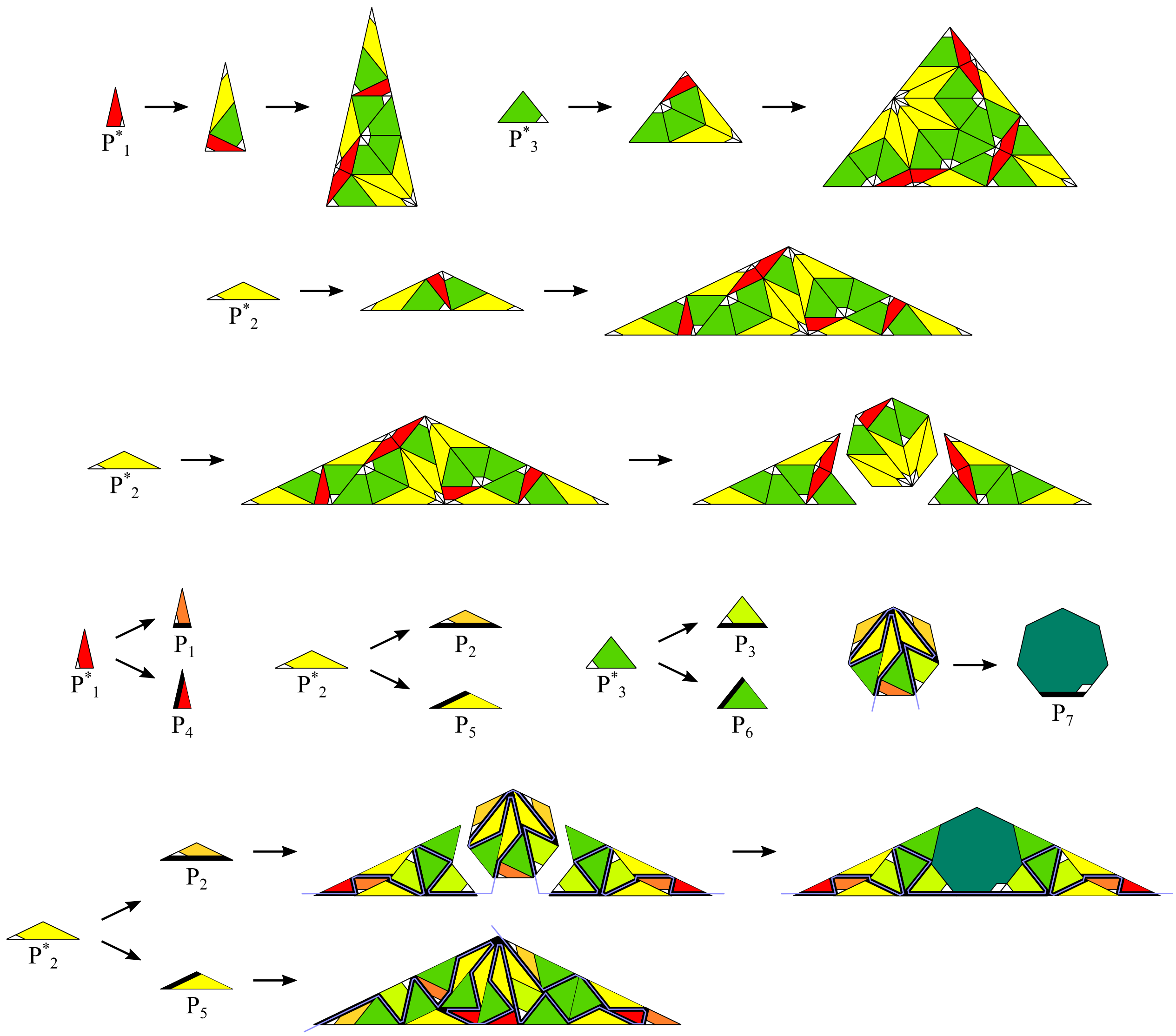

Figure 2: *The figure illustrates how to derive the space filling curve of the heptagon in Figure 3 from a cyclotomic aperiodic substitution tiling. The first two lines show the initial substitution rules of prototiles P_1^*, P_2^* and P_3^* as shown in in [15, Figure 5]. The third line shows exemplary the rearrangement of the substitution rule of prototile P_2^* so that it yields a patch in the shape of a heptagon. The fourth line shows, how the prototiles of the initial aperiodic substitution rules P_1^*, P_2^* and P_3^* can be replaced by prototiles P_1, P_2 ... P_7 with decorations. Finally the last part shows the derivation of the substitution rules of prototiles P_2 and P_5, each with a polygonal chain connecting two corner points. The blue lines were added to illustrate the sequence of the line segments. The complete set of prototiles and substitution rules is shown in Figure 3.*

substitution tiling described in [11] or [15, Chapter 4] with odd $n \geqq 5$.

In a second step we apply the decorations on the prototiles so that two corner points are connected. For every prototile in the shape of an isosceles triangle two variants are needed as illustrated in the fourth line of Figure 2. In the first variant we connect the corner points on the base, so that the line is running along the base of the prototiles, as it is the case for P_1, P_2 and P_3. In the second variant we connect the apex with one of the other corner points on the base, so the line is running along a leg of the isosceles triangle, as it is the case for P_4, P_5 and P_6. Furthermore prototiles with decorations are combined to form a patch in the shape of a regular heptagon, so that the decorations on the tiles form an uninterrupted polygonal chain. The heptagonal patch is used as an additional prototile P_7.

In a third step we rearrange the substitution rule of each prototile so that the curve on it is still running from the apex to a base corner or from base corner to base corner without interruption. In substitution rules of prototiles the curves can run along the base of the triangle but can only touch the other sides of the triangle at single points. Respectively in substitution rules of the prototiles P_4, P_5 and P_6 the curve may run along one leg but can only touch the other leg and the base at single points as well. Moreover, within the substitution rule the curve may also meet itself on single points only. This procedure will ensure that the curve in general touches itself only at single points. The result of this rearangement is shown exemplarly for prototile P_2^* on the bottom of Figure 2. The complete set of prototiles and corresponding substitution rules of the aperiodic substitution tiling with a inflation multiplier of $\frac{sin(3\pi/7)^2}{sin(\pi/7)^2} \approx 5.049$ is shown in Figure 3.

Self-similarity implies that the relative frequencies of all prototiles should be constant at all scales, so to say scale-invariant. As a consequence in a last step we have to make sure that the aperiodic substitution tilings yield “primitivity”. That means if we apply repeated substitutions to each of the singular prototiles of the substitution tiling, then sooner or later all types of prototiles will appear in each of the corresponding level-n super tiles. This is illustrated in Figure 4 for the case $n = 5$. After three substitutions all types of prototiles can be found within all types of level-3 supertiles. For the case $n = 7$ in Figure 3 it can be shown, that it takes two substitutions to demonstrate primitivity. If the condition of primitivity is not met, some types of prototiles will necessarily be thinned out during the substitutions, and this would contradict self-similarity.

To the surprise of the author the curve of the regular pentagon ($n = 5$) as shown in Figures 1 and 4 was more difficult to derive. Here a variant of the Robinson-Penrose Tiling as discussed in [8, Figure 10.3.14] and [2, Example 6.1] served as starting point. It turned out to be very difficult to modify the tiling successfully. After the third step in particular did not lead to useful results, the entry and exit points of the decorations were shifted away from the apex and from the corner points of the prototiles and of the corresponding substitution rules. This change led to finding a curve with the desired properties and self-avoidance. Its inflation multiplier is $\frac{sin(2\pi/5)}{sin(\pi/5)} = \frac{1+\sqrt{5}}{2} = \varphi \approx 1.618$ which equals the golden ratio.

Remark. For self-similarity AND self-avoidance it is very important that both are adjusted, prototiles AND substitution rules. In literature many examples can be found where a subsequent adjustment leads to a self-avoiding curve but without self-similarity.

Summary

We sketched a method to derive space-filling, self-similar curves of regular n-gons from decorations on cyclotomic aperiodic substitution tilings which are stone inflations. After some practice the method is relatively easy to handle, except that the complexity of the results increases non-proportionally with growing n. The challenging part is to identify space-filling curves for n-gons which are self-similar and self-avoiding at the same time. The method generates interesting curves with great aesthetic appeal beyond the borders of mathematical research.

In principle the desired curves can be found for arbitrary n-gons, as long as an appropriate substitution

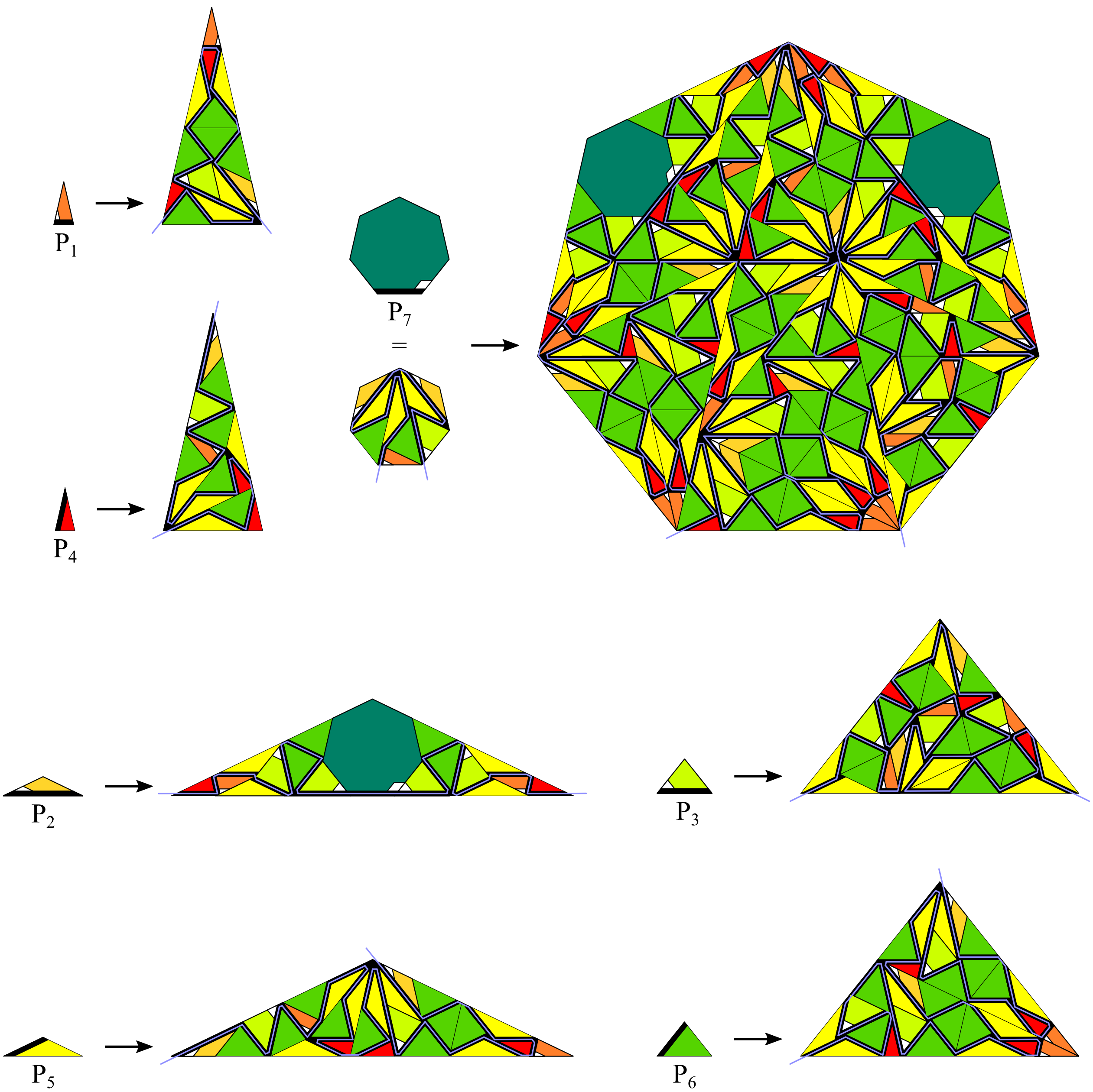

Figure 3: *Prototiles and substitution rules (level-1 supertiles) of a cyclotomic aperiodic substitution tiling. The small white triangles in prototiles P_1, P_2, P_3 and P_7 mark the chirality of the substitution rules. Each prototile carries a decoration in form of a black line on one of its sides. The black lines on the level-1 supertile of prototile P_7 form a polygonal chain which represent the first iteration of a space-filling, simple and self-similar curve of the heptagon. The blue lines were added to illustrate the sequence of the line segments.*

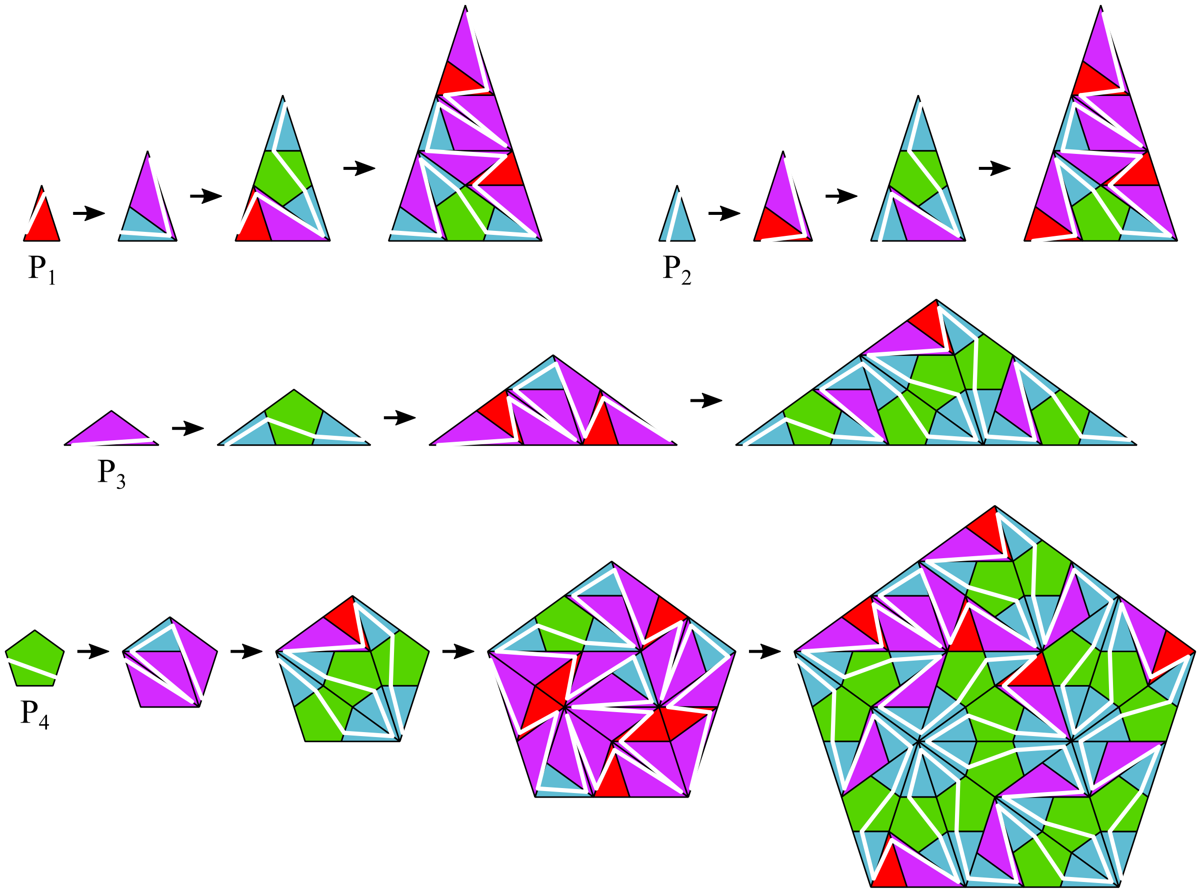

Figure 4: *Prototiles, substitution rules and some level-n supertiles of a cyclotomic aperiodic substitution tiling. Each prototile carries a decoration in form of a white line. The white lines on the level-n supertiles of prototile P_4 form polygonal chains which represent the first n iterations of an FASS curve of the pentagon. The 6th iteration of the curve is shown in Figure 1.*

tiling with a prototile in the shape of the n-gon exists. However, a complete algorithm to identify space-filling, self-similar curves or even better FASS curves for regular n-gon is not available yet and requires further research.

Acknowledgment

The author dedicates this paper to Richard David James on the occasion of his 50th birthday.

He would like to thank Dirk Frettlöh (Bielefeld University), Christian Mayr (Technische Universität Dresden), Klaus-Peter Nischke, Asta Richter (Technical University of Applied Sciences Wildau), Christian Richter (Friedrich Schiller University Jena) and Jeffrey Ventrella for the helpful discussions and comments.

References

[1] J. Arndt and J. Handl. "Puzzling Plane-Filling Curves." *Proceedings of Bridges 2019: Mathematics, Art, Music, Architecture, Education, Culture*, Linz, Austria, July 16–20, 2019, pp. 131–138. http://archive.bridgesmathart.org/2019/bridges2019-131.html

[2] M. Baake and U. Grimm. *Aperiodic Order. Vol 1. A Mathematical Invitation*, Cambridge University Press, 2013.

[3] M. Bader. *Space-Filling Curves - An Introduction with Applications in Scientific Computing*, Springer-Verlag, 2013.

[4] F. P. M. Beenker. “Algebraic theory of non-periodic tilings of the plane by two simple building blocks: a square and a rhombus..” *TH- Report (TU Eindhoven)* vol. 82-WSK04, 1982.

[5] R. Berger. “The Undecidability of the Domino Problem.” *Memoirs of the American Mathematical Society*, vol. 66, no. 1, 1966, pp. 1–72.

[6] M. Gardner. “Mathematical Games.” *Scientific American*, vol. 234, no. 6, 1976, pp. 120–125.

[7] M. Gardner. “Extraordinary Nonperiodic Tiling that Enriches the Theory of Tiles.” *Scientific American*, vol. 236, 1977, pp. 110–119.

[8] B. Grünbaum and G. C. Shephard. *Tilings and Patterns*, W. H. Freeman & Co., 1987.

[9] R. Hassell. “A Plane-Filling Curve Using Ammann A5 Tiles.” *Proceedings of Bridges 2014: Mathematics, Music, Art, Architecture, Culture*, Phoenix, Arizona, USA, July 16–20, 2014, pp. 389–392. http://archive.bridgesmathart.org/2014/bridges2014-389.html

[10] F. Henle. “Space-Filling Curves on Non-Periodic Tilings.” *G4G12 Exchange Book*, Atlanta, Georgia, USA, Mar. 30 – Apr. 03, 2016, pp. 45–51. https://www.gathering4gardner.org/g4g12-exchange-book/

[11] T. Hibma. *Half Rhombs*. Accessed on 23 January 2021. https://www.aperiodictiling.org/wpaperiodictiling/index.php/fractional-rhomb-tilings/half-rhombs/

[12] D. Hilbert. “Ueber die stetige Abbildung einer Line auf ein Flächenstück.” *Mathematische Annalen*, vol. 38, no. 3, 1891, pp. 459–460.

[13] J. Kari and M. Rissanen. “Sub Rosa, A System of Quasiperiodic Rhombic Substitution Tilings with n-Fold Rotational Symmetry.” *Discrete & Computational Geometry*, vol. 55, no. 4, 2016, pp. 972–996.

[14] B. B. Mandelbrot. *The Fractal Geometry of Nature*, W. H. Freeman & Co., 1977.

[15] S. Pautze. “Cyclotomic Aperiodic Substitution Tilings." *Symmetry*, vol. 9, no. 2:19, 2017, pp. 1–41.

[16] G. Peano. “Sur une courbe, qui remplit toute une aire plane." *Mathematische Annalen*, vol. 36, no. 1, 1890, pp. 157–160.

[17] R. Penrose. “The Role of Aesthetics in Pure and Applied Mathematical Research." *Bulletin of the Institute of Mathematics and its Applications*, vol. 10, 1974, pp. 266–271.

[18] R. Penrose. “Pentaplexity A Class of Non-Periodic Tilings of the Plane." *The Mathematical Intelligencer*, vol. 2, no. 1, 1979, pp. 32–37.

[19] G. Polya. “Über eine Peanosche Kurve." *Bull. Acad. Sci. Cracovie (Sci. Math. et nat. Serie A)*, 1913, pp. 305–313.

[20] E. A. Robinson Jr. “The Tilings of Kari and Culik." 2009. https://blogs.gwu.edu/robinson/leftovers/

[21] G. Teachout. *Fractal Space Filling Curves*. Accessed on 14 January 2021. http://teachout1.net/village/fill.html

[22] J. Ventrella. *The Family Tree of Fractal Curves*, Eyebrain Books, 2019.

[23] J. Ventrella. “Portraits from the Family Tree of Plane-filling Curves.” *Proceedings of Bridges 2019: Mathematics, Art, Music, Architecture, Education, Culture*, Linz, Austria, July 16–20, 2019, pp. 128–130. http://archive.bridgesmathart.org/2019/bridges2019-123.html