# Pretty 3D Polygons: Exploration and Proofs

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## Abstract

We explore 'pretty' 3D (skew) polygons and prove their existence. These generalize the well-known regular 2D polygons. In 3D, an additional regularity condition is imposed: all *edge torsion angles* must be equal in absolute value. The torsion angle of an edge is the dihedral angle between the planes spanned by the edge and each of its two adjacent edges. We define an infinite family of pretty 3D polygons with both rotation and reflection symmetries. This resolves an open problem about the existence of certain pretty 3D polygons. Moreover we present some ad hoc specimens, including two trefoil knots, that do not have reflection symmetry. Finally, we present some pretty 3D polygons that can be morphed while preserving their prettiness.


## Introduction

Koos Verhoeff [4] was a mathematical artist known in particular for his sculptures based on closed spatial paths constructed from beams with a polygonal cross section connected by miter joints (Fig. 1). His design challenge for such paths was to ensure that the longitudinal edges of the beams properly connect across all joints [5]. It is easy to make all-but-one of the connections proper miter joints: just start with a segment, and repeatedly connect the next segment with a proper miter joint. The last joint that closes the loop, however, will generally not be proper, because the cross section will have accumulated a rotation, also known as torsion (defined below). In order for the last connection to be a proper miter joint, the total torsion angle must be a rotational symmetry of the cross section (e.g., a multiple of 90° in case of a square cross section).

Koos used various techniques to control the torsion. Let's first define this torsion. The torsion angle of edge $(U, V)$ in a polygonal path $\ldots, T, U, V, W, \ldots$ is the directed dihedral angle between the plane passing through $(T, U, V)$ and the plane passing through $(U, V, W)$, where the angle's sign is determined by the right-hand rule when rotating plane $(T, U, V)$ toward plane $(U, V, W)$ about vector $(U, V)$. The total torsion of a closed polygonal path is the sum of the (directed) torsion angles over all edges of the path.

**Figure 1:** *Design types in sculptures by Koos Verhoeff (left to right): Bicolored Torus Path (ad hoc, wood, 50cm); Ovonde (FCC lattice, stainless steel, 3m);* $(---+++)^4$ *(constant torsion, bronze, 5cm)*

Initially, Koos approached torsion control in an *ad hoc* way (Fig 1, left), where he tinkered with the positions of the points, preserving certain design parameters such as symmetries, until 'it worked' [5]. Next, he restricted the paths to *lattices* (Fig 1, center), where joint angles and torsion angles are severely constrained. A particularly elegant approach uses *skew* miter joint, where (unlike *regular* miter joints) the cut face does not lie in the bisector plane of the joint angle [5, 9]. Finally, Koos used paths with *constant torsion* (Fig 1, right), where all torsion angles are equal in absolute value and correspond to a rotational symmetry of the beam's cross section [6].

The designs described in [6], however, are based on numerical approximations of 'solutions' whose mathematical existence was not proven there. When those approximate 'solutions' are constructed in wood, they are sufficiently precise that we do not notice the issue. We did have confidence in their existence, but a proof has eluded us for a long time. In this paper, we prove the existence of such 'pretty' constant-torsion polygons. There is some overlap with [6], but the next section makes the paper self-contained.

## Definitions

What does it mean that a closed polygonal path in 3D is pretty? This is easy to explain by extending 2D turtle geometry [1] to 3D, following [3]. At each moment in time, the *state* of the 3D turtle is captured by its *position*, a vector $v$, and two orthogonal unit vectors $h$ (*heading*) and $n$ (*normal*). Its *initial* state is given by $v, h, n := (0, 0, 0),\ (1, 0, 0),\ (0, 0, 1)$. For convenience, its *port* (left-side) vector $p$ is defined by $p = n \times h$ (cross product), so that vectors $h, p, n$ form a right-handed orthonormal system (initially, $p = (0, 1, 0)$). The 3D turtle obeys the following three commands.

- $M_d$ (*move* along $h$ by distance $d$): $v := v + d\, h$
- $T_\varphi$ (*turn* about $n$ by angle $\varphi$): $h := h \cos\varphi + p \sin\varphi$
- $R_\psi$ (*roll* about $h$ by angle $\psi$): $n := n \cos\psi - p \sin\psi$

A turtle *program* is a sequence of turtle commands. A program is called *closed* when final and initial positions are equal. It is called *properly closed* when final and initial states are equal. For convenience, we introduce the abbreviation $S\,(d, \psi, \varphi)$, pronounced *segment*, defined by $S\,(d, \psi, \varphi) = M_d R_\psi T_\varphi$ (note the order: roll occurs before turn). The *path* of a turtle program consists of all the points it visits.

As proven in [3], each properly closed turtle program is congruent to a turtle program of the form $q = S\,(d_1, \psi_1, \varphi_1)\ \ldots\ S\,(d_k, \psi_k, \varphi_k)$ visiting vertices $v_0 = (0, 0, 0), v_1, \ldots, v_{k-1}, v_k = v_0$, where

- $d_i > 0$
- $-180° < \psi_i \le 180°$ (these roll angles are the edge torsion angles)
- $0 < \varphi_i < 180°$ (negative $\varphi_i$ are not needed; just roll the turtle upside down)
- $v_1$ lies on the open $x^+$-half-line, and
- $v_{k-1}$ lies in the open $(x, y^+)$-half-plane.

In [6], regular 2D polygons are generalized to 3D by imposing the following conditions on program $q$ above.

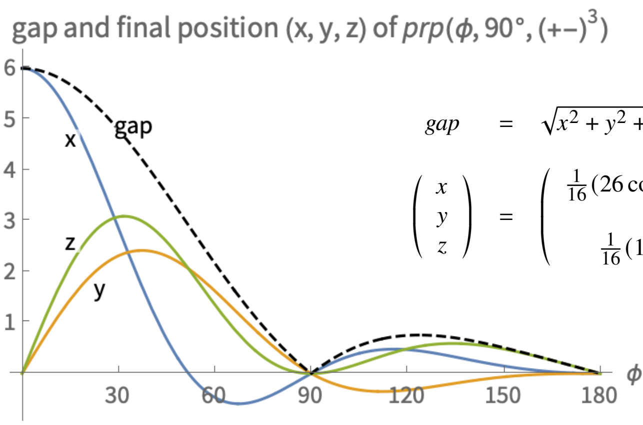
1. All $d_i$ are equal (a.k.a. *equilateral*), and without loss of generality $d_i = 1$ ($d_i$ is just a scaling factor).
2. All $\varphi_i$ are equal (a.k.a. *equiangular*).
3. All $|\psi_i|$ are equal (we call this *equitorsal*; without this condition, too much is possible).

We will call these *pretty* 3D polygons. For $\psi_i = 0$, the path is a regular 2D polygon. For $\psi_i \neq 0$ and all roll signs equal, $q$ generates a discrete helix (zigzag if $\psi_i = 180°$) that cannot close. The absolute value in the roll angle condition was introduced to allow interesting closed designs. Later, we will similarly relax the turn angle condition. Observe that when constructing a pretty 3D polygon with round beams and miter joints, all the pieces are congruent (possibly mirror images). Our research question is: For what parameter combinations do non-planar pretty (skew) polygons exist, preferably with nontrivial rotation symmetries? In the conclusion, we relate this to *regular skew polygons*.

## Sufficient condition: an infinite family of pretty 3D polygons

Given turn angle $\varphi > 0$, absolute roll angle $\psi \geq 0$, and sequence $s$ of roll angle signs (+ or −), we denote by $prp\,(\varphi, \psi, s)$ (pretty polyline) the 3D turtle program $S\,(1, s_1\psi, \varphi)\ S\,(1, s_2\psi, \varphi)\ \ldots\ S\,(1, s_k\psi, \varphi)$. This program generates a pretty polygon *provided it is properly closed.* For instance, $prp\,\left(90°, 90°, (+-)^3\right)$ is a pretty polygon on the cube, visiting six vertices, avoiding two diametrically opposite vertices, also known as a zigzag polygon or antiprismatic skew polygon.

Koos Verhoeff explored various sign patterns for $\psi = 90°$, some of which are described in [6]. He considered the function $gap_{\psi,s}\,(\varphi)$ that maps turn angle $\varphi$ to the distance between the initial and final position of program $prp\,(\varphi, \psi, s)$. Observe that *gap* is a continuous function, since it is a (complicated) algebraic expression in (co)sines (Fig. 2, right). By plotting its graph you can see where it might have zeros (Fig. 2, left). But in most cases it is not clear that these are true zeros, because the continuous function *gap* is non-negative, and hence we cannot apply the Intermediate Value Theorem. Phrased differently, the three position coordinates need to be zero *simultaneously*. Koos constructed his sculptures from numerical approximations of those conjectured zeros. Moreover, note that a zero gap implies closure but not necessarily proper closure, as illustrated in [6].




$$gap = \sqrt{x^2 + y^2 + z^2} \quad \text{where}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{16}(26\cos\varphi + 24\cos 2\varphi + 21\cos 3\varphi + 8\cos 4\varphi + \cos 5\varphi + 16) \\ \frac{1}{8}(6\sin\varphi + 10\sin 2\varphi + 6\sin 3\varphi + \sin 4\varphi) \\ \frac{1}{16}(18\sin\varphi + 16\sin 2\varphi + 19\sin 3\varphi + 8\sin 4\varphi + \sin 5\varphi) \end{pmatrix}$$

**Figure 2:** *Left: gap and final position $(x, y, z)$ of prp $\left(\varphi, 90°, (+-)^3\right)$; right: formula for final position as function of $\varphi$, obtained via Mathematica*

The designs of Koos involve sign sequences that are three, four, or five repetitions of a *motif* (the period). His motifs also seem to have some internal structure (more about this in a moment). The resulting pretty polygons have rotations and reflections as symmetries. The key ingredient in understanding these designs turns out to be the reflection symmetries. To define the relevant structure of the motifs, we introduce three auxiliary functions on sign sequences:

$$\begin{aligned} rev(s) &= s \text{ with all roll signs in reverse order} \\ flip(s) &= s \text{ with all roll signs flipped } (+ \leftrightarrow -) \\ refl(s) &= rev(flip(s)) \qquad \text{(this is the same as } flip(rev(s))) \end{aligned}$$

Functions *rev*, *flip*, and *refl* are *involutions*, i.e., their own inverse.

Koos' motifs $s$ have the property $s = refl(s)$, which we call *refl* symmetric. Examples from [6] are: +−, ++−−, +++−−−, ++−++−−+−−, and +++−−++−−−. Note that $s = refl(s)$ if and only if there exists a sign sequence $t$ such that $s = t\ refl(t)$, where juxtaposition denotes concatenation of sequences. We call such a $t$ a *base* of the motif. Note that motifs for which base $t$ is reversal symmetric (a palindrome, i.e., $t = rev(t)$), generate pretty polygons with an additional *rotoreflection* symmetry, as illustrated in Fig. 1 (right). Also observe that since a motif is to be repeated cyclicly, cyclic rotations of a motif are equivalent (give rise to congruent paths). E.g., motif ++−− is equivalent to +−−+, −−++, and −++−, but only two of these are *refl* symmetric. Also the reversal of a motif will generate a congruent path.

We will prove that for any *refl*-symmetric motif $s$ (or equivalent motif) and any roll angle $\psi$, there exist infinitely many numbers $k$ and non-trivial turn angles $\varphi$ such that $prp\left(\varphi, \psi, s^k\right)$ is a pretty polygon. In fact, we will prove something more general. We generalize *rev*, *flip*, and *refl* to arbitrary turtle programs $q$:

$$\begin{aligned} rev(q) &= q \text{ with its move/turn/roll commands in reverse order} \\ flip(q) &= q \text{ with all its roll signs flipped } (R_\psi \mapsto R_{-\psi}) \\ refl(q) &= rev(flip(q)) \qquad \text{(this is the same as } flip(rev(q))) \end{aligned}$$

These generalized functions are also involutions.

**Reflection Lemma** Let $q$ be a turtle program, and let $v'$ and $h'$ be the position and heading after $q$. Program $r = q\ refl(q)$ generates a path that has a reflection symmetry, with reflection plane passing through $v'$ and perpendicular to $h'$. Let $v'', h'', p'', n''$ be the final state after $r$. Then $v'', -h'', p'', n''$ is the reflection of the initial turtle state. Fig. 3 illustrates this for $q = T_{30^\circ}\ S\,(1, 60^\circ, 90^\circ)\ \ S\,(2, 111.625^\circ, 60^\circ)$. Hence, $refl(q) = T_{60^\circ}\ R_{-111.625^\circ}\ M_2\ T_{90^\circ}\ R_{-60^\circ}\ M_1\ T_{30^\circ}$. For details see the supplementary material.

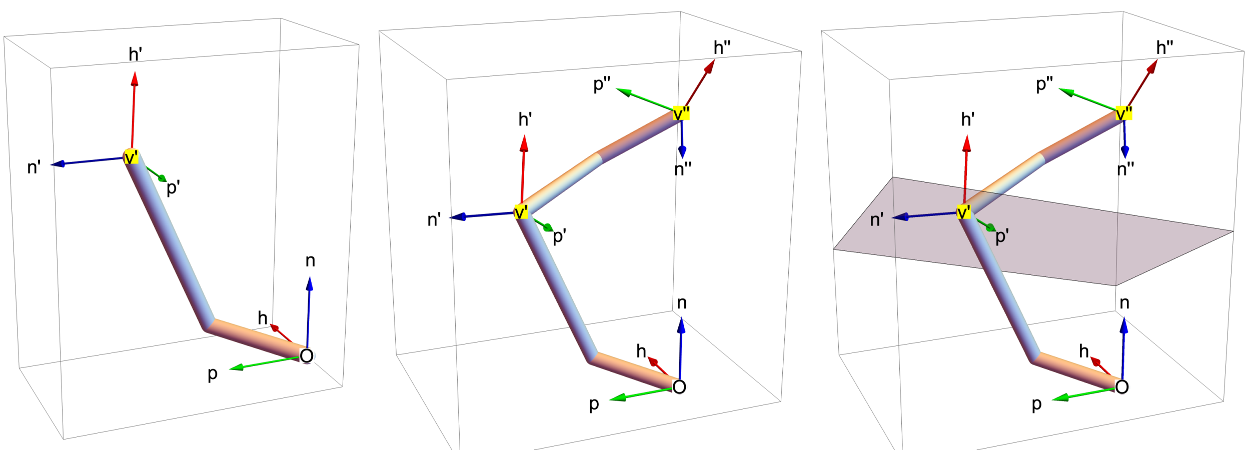



**Figure 3:** *Reflection Lemma: path of q (left); path of r (center); path of r with reflection plane (right)*

**Repeated Reflection Theorem** Let $q$ be a turtle program, and let $r = q\ refl(q)$. Consider program $r$ executed $k$ times, i.e., $r^k = q\ refl(q)\ q\ refl(q)\ \ldots$. Since $refl(refl(q)) = q$, we can repeatedly apply the Reflection Lemma, and we see that the path generated by $r^k$ consists of $k$ copies of the $r$-path rotated about the intersection of the reflection planes for $q\ refl(q)$ and for $refl(q)\ refl(refl(q)) = refl(q)\ q$. The rotation angle is twice the angle between these reflection planes. Since the angle $\theta$ between the reflection planes is a continuous function of each of the command parameters in $q$, we can vary (one of) those parameters to make the rotation angle $2\theta$ a fractional multiple of 360°, i.e., $\theta = \frac{a}{b}360^\circ$. Consequently, the program $r^b$ produces a path that wraps $a$ times around the rotation axis and then closes properly.

Fig. 4 illustrates this for $q, r$ defined above, where the scene has been rotated so that the rotation axis is vertical. The angle 111.625° was chosen such that the angle between the reflection planes equals 60°. This means that three repetitions gives a properly closed path, thus a pretty polygon, with threefold rotation symmetry and reflection symmetry.

**Corollary** We now apply this to the program $prp\,(\varphi, \psi, s)$ where $s = t\ refl(t)$ for some (roll) sign sequence $t$. In order to apply the Repeated Reflection Theorem, we need to find an appropriate ‘half

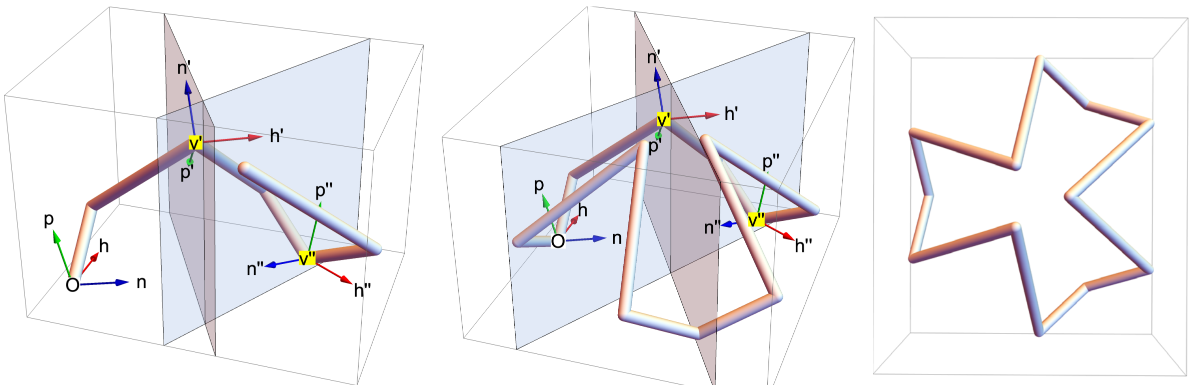



**Figure 4:** *Repeated Reflection Theorem: q refl(q) q and 2 mirrors (left); $r^3$ (center); $r^3$ top view (right)*

period' $q$. It is not $prp\,(\varphi, \psi, t)$, because $prp\,(\varphi, \psi, \mathit{refl}(t)) \neq \mathit{refl}(prp\,(\varphi, \psi, t))$. From [3], we know that in a properly closed program, we can cyclicly shift the commands, without affecting the shape of the path. Moreover, observe that $T_\varphi = T_{\varphi/2}\ T_{\varphi/2}$. Let $prp\,(\varphi, \psi, t) = M_1\ R_{\pm\psi}\ T_\varphi\ \ldots\ M_1\ R_{\pm\psi}\ T_\varphi$. Next, define $q = T_{\varphi/2}\ M_1\ R_{\pm\psi}\ T_\varphi\ \ldots M_1\ R_{\pm\psi}\ T_{\varphi/2} = T_{\varphi/2}\ prp\,(\varphi, \psi, t)\ T_{-\varphi/2}$, that is, we have cycled half of the trailing turn command to the front. We now have that $prp\,(\varphi, \psi, s)^k$ is properly closed if and only if $q^k$ is properly closed. Finally, observe that $\mathit{refl}(q) = T_{\varphi/2}\ R_{\mp\psi}\ M_1\ T_\varphi\ \ldots R_{\mp\psi}\ M_1\ T_{\varphi/2} = T_{\varphi/2}\ M_1\ R_{\mp\psi}\ T_\varphi\ \ldots M_1\ R_{\mp\psi}\ T_{\varphi/2} = T_{\varphi/2}\ prp\,(\varphi, \psi, \mathit{refl}(t))\, T_{-\varphi/2}$, because $M_d\ R_\psi = R_\psi\ M_d$, i.e., rotation and translation about/along the same axis commute. So, we can apply the Repeated Reflection Theorem to $q$. Consequently, for $k$ sufficiently large, there exist $\varphi$ and $\psi$ for which $prp\,(\varphi, \psi, s)^k$ is properly closed.

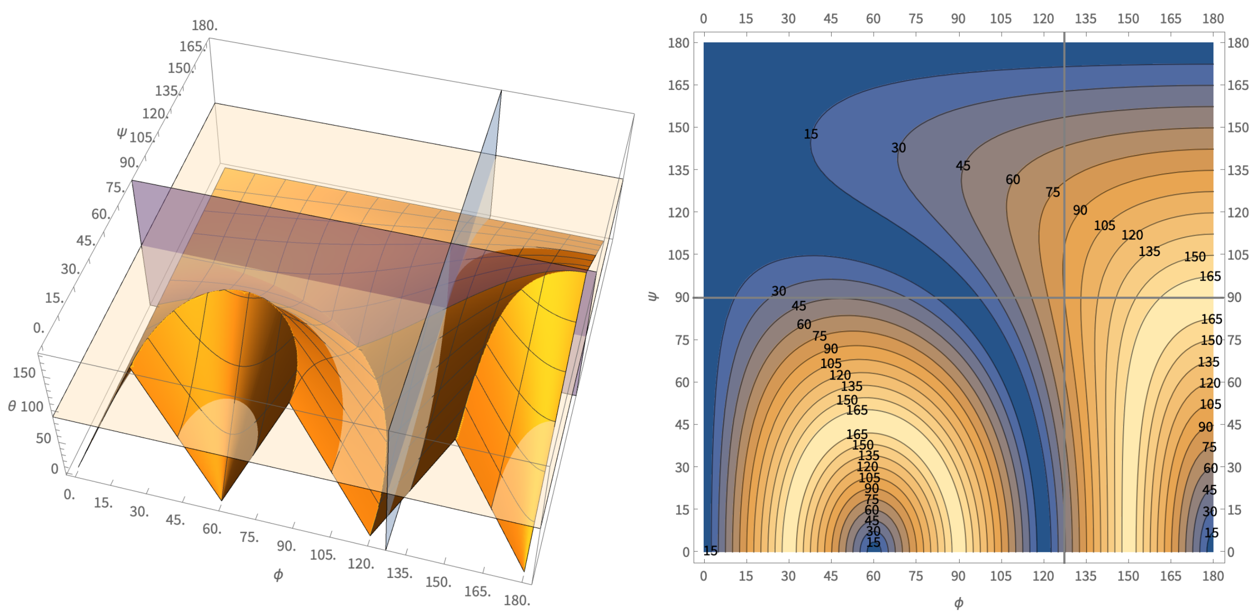



**Figure 5:** *Plot of rotation angle $|\theta|$ as function of $\varphi, \psi$ for prp $(\varphi, \psi, +{+}{+}{-}{-}{-})$ (left: 3D; right: contour)*

Fig. 5 shows the rotation angle $\theta$ as function of turn/roll angles $\varphi, \psi$ for roll signs $s = +{+}{+}{-}{-}{-}$. For $\varphi, \psi = 127.176°, 90°$ (marked by cross hair), the rotation angle is 90°. The exact formula for $\psi = 90°$ obtained via Mathematica is $\cos\frac{\theta}{2} = \frac{1}{8}(6 + 3\cos\phi - 2\cos 2\phi + \cos 3\phi)$. The resulting pretty polygon is shown in Fig. 1 (right) with a square beam.

## Other Solutions

Besides the infinite family of pretty polygons described in the preceding section, there are also other constructions. Consider turtle program $q$. We now study properties of $q^k$, i.e., $q$ repeated $k$ times. According to Chasles' Theorem the final state after $q$ can be obtained by a single *screw operation* applied to the initial state. That is, there exist a (directed) rotation axis (say given by unit vector $u$ and point $w$), a (signed) rotation angle (say $\theta$), and a (signed) translation distance along that same axis (say $c$), such that the final state is obtained by rotating the initial state about $u$ at $w$ over angle $\theta$ and translating it by $cu$ (see Fig. 6, left). Therefore, program $q^k$ forms a helix around the axis. Compare this to the Looping Lemma of [1] for 2D turtle geometry.

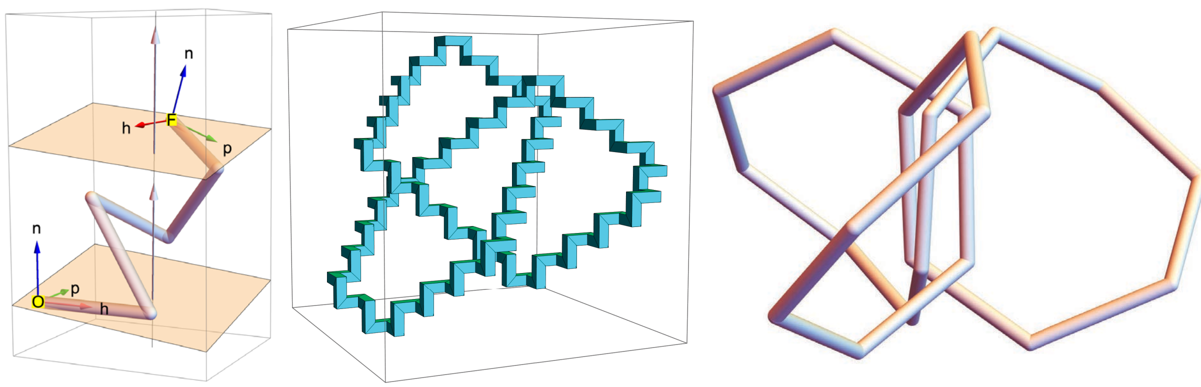



**Figure 6:** *Chasles' Theorem applied to prp* $(120°, 90°, +{-}{-}{-}+)$ *(left); trefoil knot in the SC lattice prp* $\left(90°, 90°, +^5{-}^{11}+^{17}{-}^{11}\right)^3$ *(middle); trefoil knot prp* $(58.82°, 32.07°, ++{-}{-}{-}{-}{-}{-})^3$ *(right)*

Program $q^k$ is properly closed when $q$'s translation distance $c = 0$ and simultaneously $q$'s rotation angle $\theta$ is a multiple of $360°/k$. Thus, we know when *prp* $(\varphi, \psi, s)^k$ is a pretty polygon with an order $k$ rotation symmetry. The question is whether such $q^k$ exist. Fig. 6 (middle) shows an example: *prp* $\left(90°, 90°, +^5{-}^{11}+^{17}{-}^{11}\right)^3$ is a pretty trefoil knot in the SC lattice, where the helix axes are in BCC [7, 8].

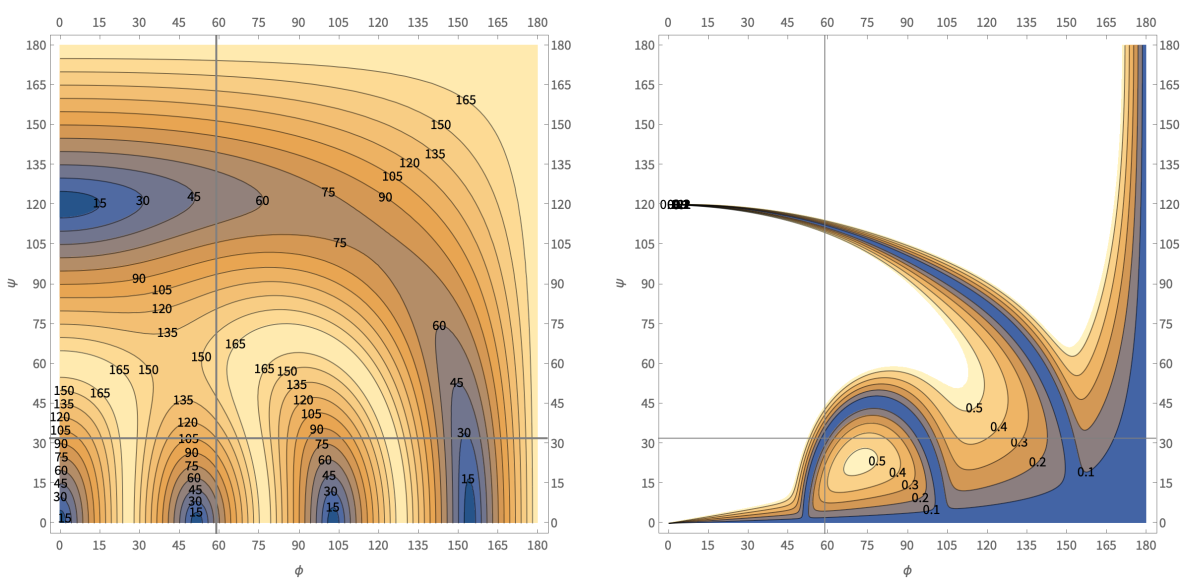



**Figure 7:** *Contour plots for rotation angle* $|\theta|$ *(left) and* $(0.5$*-clipped) translation distance c (right) for prp* $(\varphi, \psi, ++{-}{-}{-}{-}{-}{-})$*; in both plots, point* $\varphi, \psi \approx 58.82°, 32.07°$ *is marked by a cross hair*

For *refl*-symmetric roll sign sequences $s$, the translation distance equals 0 independent of $\varphi, \psi$. We have not found a general construction for arbitrary $s$, but we now show that sporadic solutions exist. Consider $q = prp\,(\varphi, \psi, {+}{+}{-}{-}{-}{-}{-}{-})$. Fig. 7 shows contour plots for rotation angle $\theta$ (left) and translation distance $c$ (right) for this program $q$. By continuity of the contour lines, there exists at least one exact intersection point for the contours of $\theta = 120°$ and $c = 0$. For $\varphi, \psi \approx 58.82°, 32.07°$, the rotation angle $\theta \approx 120°$ and translation distance $c \approx 0$ (marked by cross hairs). The resulting pretty polygon $q^3$ turns out to be a(nother) trefoil knot (Fig. 6, right). It has 21 edges, and order-2 and order-3 rotation symmetry, but no reflection symmetry.

Pretty polygons with a *refl*-symmetric roll sign pattern can be morphed by continuously varying their turn and roll angles, such that the angle $\theta/2$ between the reflection planes is constant; i.e., by following a contour line with constant $\theta$ (cf. Fig. 5 and 8 left). The supplementary material shows an animation of such morphing for $s = ({+}{+}{+}{-}{-}{-})^4$ for $\theta = 90°$ and $s = ({+}{+}{-}{+}{+}{-}{-}{+}{-}{-})^2$ for $\theta = 180°$. The latter resembles the folding of a closed strip, from a single loop (winding number 1) to a triple loop (winding number 3).

## Varying the signs of the turn angles

Instead of keeping the turn angles fixed and varying the signs of the roll angles, it is also possible to investigate 3D polygons where the turn angle signs vary and the roll angles are fixed. Equivalently, the turn angle signs are fixed, but the roll angles are either $\psi$ or $\psi - 180°$ (for $\psi = 90°$, this reduces to positive/negative roll angles). For instance, paths in the *triamond* (a.k.a. K4, $(10, 3)$-a, Laves) lattice have edges of unit length, turn angles of $\pm 60°$, and roll angles of $\arccos \frac{1}{3} \approx 70.5°$ (or $-\arccos \frac{1}{3}$ for the mirrored triamond lattice). The shortest cycle in the triamond lattice has length ten, with turn signs $({+}{-}{-}{-}{+})^2$. Fig. 8 (right) shows this polygon in two forms: using the *MathMaker* kit [9] and using *Bamboozle* triangles [8]. The *MathMaker* construction kit has building blocks made from $1 : \sqrt{2}$-rectangular beams cut at 45°, producing square cut faces, allowing regular miter joints at 90° and skew miter joints at 120°. There are two types of building block: trapezoids and parallelograms. When connected only by skew miter joints, the torsion angles can be restricted to $\arccos \frac{1}{3}$, thus allowing to construct all paths in the triamond lattice. The equilateral triangles in the *Bamboozle* connect at dihedral angles of $\arccos \frac{1}{3}$, and thereby act as the angle-spanning planes, in which the turtle turns.

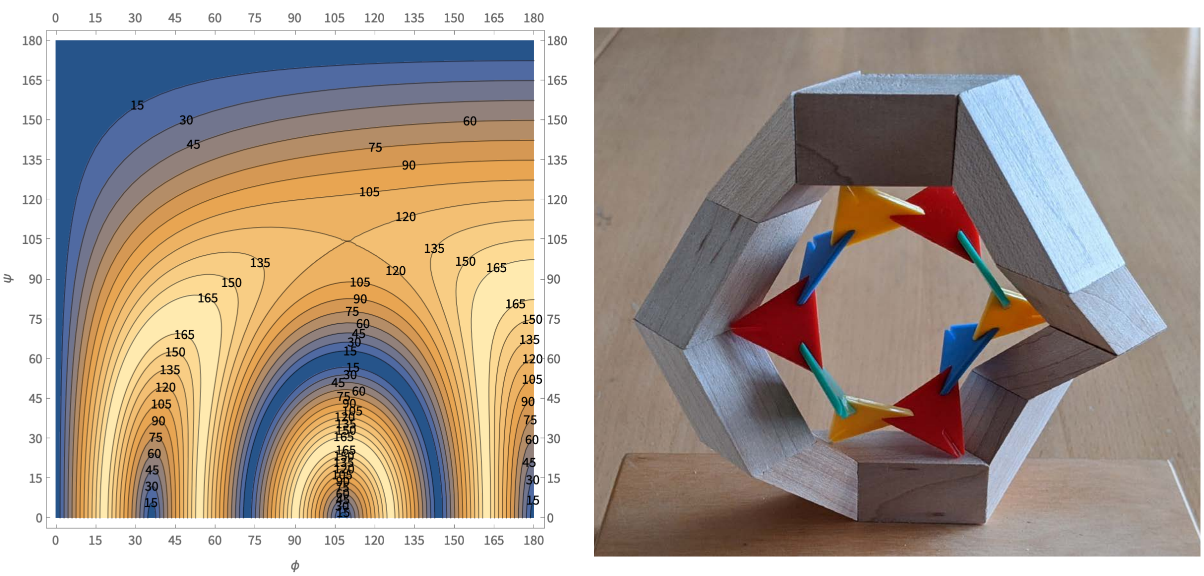



**Figure 8:** *Contour plot of $\theta$ for ${+}{+}{-}{+}{+}{-}{-}{+}{-}{-}$ (left); 10-edge skew polygon in triamond lattice with turn signs $({+}{-}{-}{-}{+})^2$, from MathMaker blocks and Bamboozle triangles (right)*

## Conclusions

We defined pretty 3D polygons as equilateral, equiangular, and equitorsal (skew) polygons. When constructing a pretty polygon from round beams and miter joints, all beam segments are congruent. We proved the existence of an infinite family of pretty polygons, including the ones designed by Koos Verhoeff, where the sequence of roll signs is *refl* symmetric. We also showed some sporadic pretty polygons. Pretty polygons based on a *refl*-symmetric roll sign sequence can be morphed while preserving prettiness.

Related to this are so-called *regular* skew polygons, which are equilateral and *vertex-transitive* (every vertex can be mapped to every other vertex by a symmetry, a.k.a. isogonal). Regular skew polygons are zigzag (antiprismatic), and hence pretty with roll sign pattern $(+-)^k$.

For future work we want to look into (avoidance of) self intersection. Another interesting question is to characterize all 3D polygons that can be constructed from round beams with miter joints, where all pieces are congruent. In this case, there could be two different turn angles that alternate. Planar polygons where turn signs vary according to a pattern are also interesting to study; e.g., $(++-)^4$ with turn angle $\varphi = 90°$ outlines a plus sign. Further details can be found in [2].

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