

Creating Non-Euclidean Art with Craft Technologies

Douglas Dunham¹ and Lisa Shier²

¹Dept. of Computer Science, Univ. of Minnesota Duluth; ddunham@d.umn.edu

²University of Maryland Global Campus; kwajshier@yahoo.com

Abstract

We have created two non-Euclidean art works, a papercrafted version of a hyperbolic pattern inspired by M.C. Escher and a free-standing lace dodecahedron. We will provide some background for each of the pieces, explain how they were developed, and show the resulting art works.

Introduction

Our goal was to create mathematical art works using craft technologies. The first work is a 16"×16" papercrafted pattern of shells in the Poincaré disk model of hyperbolic geometry, shown in Figure 1. The second work is a 6"×6"×6" free-standing lace dodecahedron decorated with Greek letters, which is shown in Figure 2.

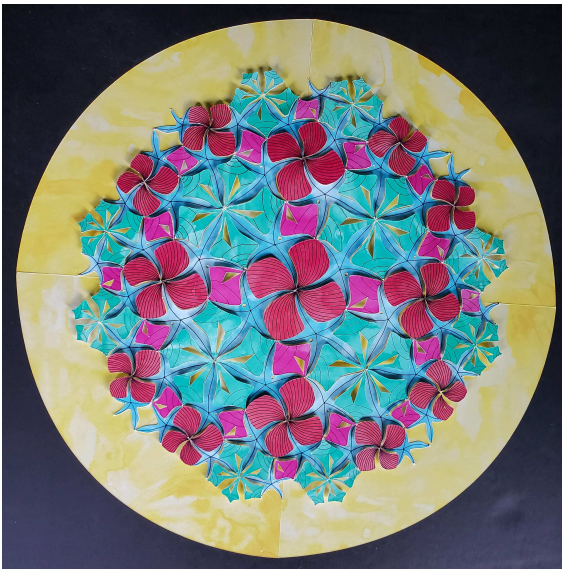


Figure 1: *The 16"×16" shell pattern.*



Figure 2: *The 6"×6"×6" lace dodecahedron.*

We have previously used a computer controlled sewing machine to create an embroidered hyperbolic fish pattern [2], and we wanted to explore other craft technologies. The first project involved using papercrafting, i.e. a paper-cutter / plotter, to create an art work. We decided on implementing a hyperbolic pattern of shells that we had previously designed. The second project also involved the computer controlled sewing machine, but to make free-standing lace instead of embroidery. The goal was to decorate each of the faces of a dodecahedron with different letters of the Greek alphabet that had mathematical significance.

In the next section, we will discuss shell patterns in general, then the construction of our shell pattern in the following section. Next, we discuss our dodecahedron. Finally, we discuss the educational use of craft technologies and indicate directions of future work.

M.C. Escher and Shell Patterns

In August 1941, the Dutch artist M.C. Escher created a repeating Euclidean pattern of shells that he numbered 42 among his Regular Division Drawings. This pattern contained red scallops, light green conchs, light brown snails, and yellow starfish, and is shown in Figure 3. This work can also be viewed on the M.C. Escher web site [3], and on page 148 of Doris Schattschneider's *M.C. Escher: Visions of Symmetry* [5]. The Dutch crystallographer, Carolina MacGillavry included it in her book *Fantasy and Symmetry: The Periodic Drawings of M.C. Escher* [4], maybe as a test for her students since the apparent 4-fold rotations at meeting points of the scallops are actually only 2-fold rotations, as will be explained below. In 1963, Escher also designed a shell pattern for icosahedral tin boxes for the Verblifa Company — meant to contain confections. Figure 4 shows one of those boxes, and Schattschneider also has a picture of one on page 295 of *M.C. Escher: Visions of Symmetry*.



Figure 3: Regular Division Drawing 42.



Figure 4: Escher's tin box 5''x5''x5''.

In 1981, Dunham and his students created a computer program that could draw repeating patterns in the Poincaré disk model of hyperbolic geometry [1]. That program could draw patterns based on the regular tessellations $\{p, q\}$ of the hyperbolic plane by regular p -sided polygons meeting q at a vertex, provided that $(p-2)(q-2) > 4$ to be hyperbolic. The program could also draw hyperbolic shell patterns like that of Figure 1, with a slight modification to take into account the 2-fold rotations at scallop meeting points.

The symmetry of Escher's Regular Division Drawing 42 is $p4$ in crystallographic notation, or 442 in orbifold notation, which suggests that a general shell pattern could have symmetry group $pq2$. Such a pattern would be spherical, Euclidean, or hyperbolic, depending on whether $(p-2)(q-2)$ is less than, equal to, or greater than 4, respectively. For convenience we label such a pattern as (p, q) , so Escher's Drawing 42 would be denoted $(4, 4)$. In these patterns there are two kinds of meeting points of conchs, with p -fold and q -fold rotational symmetry; all scallop meeting points have 2-fold rotational symmetry. Figure 5 shows the $(5, 5)$ pattern that inspired the papercrafted version of Figure 1. Figure 6 shows the same pattern, but with centered conchs instead of scallops.

In Figures 5 and 6 there are two kinds of 5-fold rotation points about the meeting points of the bottoms of the conchs: in one set of 5 conchs, the tops of the conchs touch the black openings of the brown snails; in the other set the black snail openings are away from the tops of the conchs. In Figure 5 the upper-right and lower-left groups of conchs nearest the center are of the first kind, and the upper-left and lower-right nearest

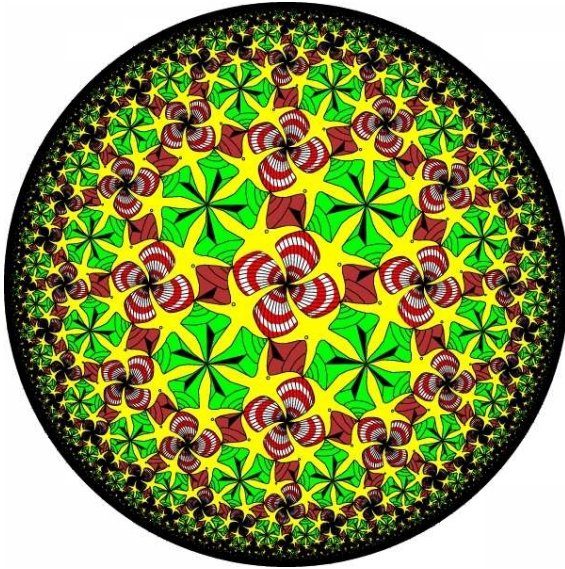


Figure 5: *The (5,5) pattern, scallop centered.*

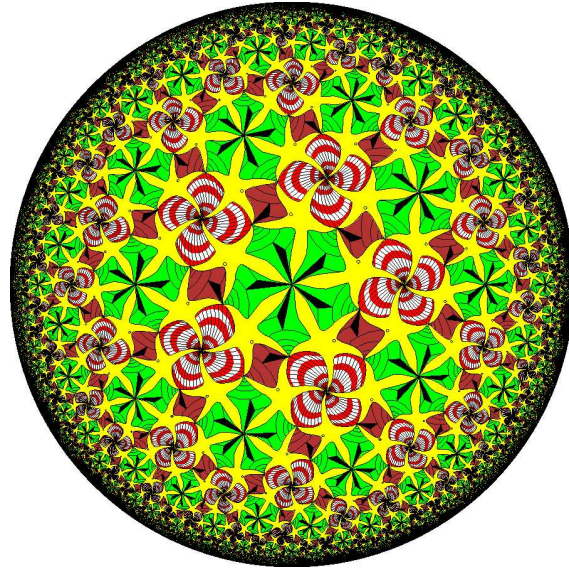


Figure 6: *The (5,5) pattern, conch centered.*

groups are of the second kind. In general, the p -fold points are of the first kind and the q -fold points are of the second kind. Those groups of conchs alternate about each of the groups of four scallops. Thus the center of each group of scallops is only a 2-fold rotation point. The same is true of Escher's Drawing 42 — there are two kinds of 4-fold meeting points of conchs, which alternate about the meeting points of the four scallops. The fact that there are two kinds of conch rotation points in a (p, q) pattern is more apparent when p and q are different, as shown in Figures 7 and 8. In Figure 7 the conchs in groups of 5 have their tops touching

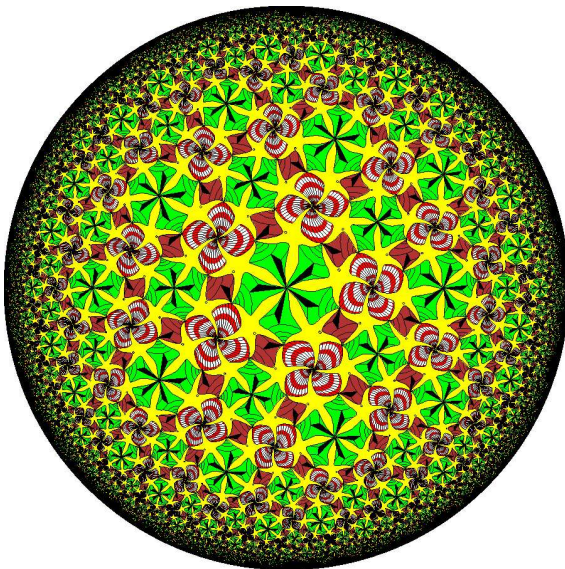


Figure 7: *The (5,4) shell pattern.*

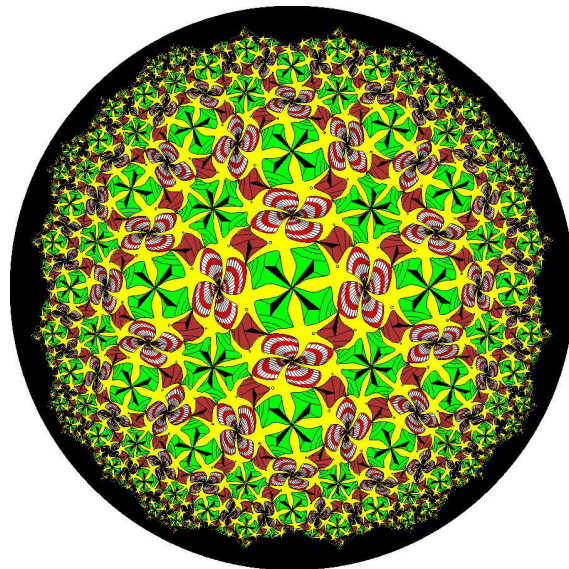


Figure 8: *The (4,5) shell pattern.*

the black openings of the snails, whereas the snails touching conchs in groups of 4 have their openings away from the tops of the conchs. In Figure 8, it is just the reverse. Also in Figure 8 it is apparent that the groups of 4 scallops only have 2-fold rotational symmetry, since the transformation of the pattern into that geometry somewhat squashes those scallop groups.

The Construction of the Hyperbolic Shell Pattern

When constructing the papercrafted artwork, one significant limitation to the size and completeness of the shell pattern was the size of the materials. Papercrafting machines have an upper size limit on the paper they can cut; this limit is typically a 12" square. Papers are available in 12"x12" and 8.5"x11" sizes. However, the type of paper we selected was only available in the smaller size. The largest piece of paper in our work, the yellow circle, was constructed out of 4 pieces, each being a quadrant of the circle. This limited the size of the circle to 16" in diameter. On the small end of the spectrum, below a certain size, the paper pieces become too hard to work with and the lines tend to blend together since the pen width is constant. This limited the total extent of the hyperbolic pattern that could be created with paper pieces. Additionally, the smaller pieces aren't exactly self-similar to the larger ones. Minor modifications needed to be made on the regions around the shell openings so that there was sufficient width of the paper for the piece to hold together.

For this project we used the Brother ScanNCut SDX225 papercrafting machine, shown in Figure 9. This machine was chosen over its rival, the Cricut Maker, because it is far more accurate at aligning the drawing and cutting, which was critical for this project. The software for the ScanNCut can take a variety of inputs, but scalable vector graphics objects (SVG files) is the most practical input for this type of project. However, this software does not provide good tools for creating the type of SVG objects we wanted. The software used to generate our SVG objects was Craft 'N Cut by Floriani, a noted producer of embroidery software. (We used a different Floriani product for our embroidery work last year [2]). The Scan 'N Cut software was used to hand digitize all of the shells and starfish in the hyperbolic image. The symmetries of the image made this task somewhat easier. However, there is no software support for the distortions of the shells in hyperbolic geometry, so for example, each of the five conchs in each cluster had to be digitized separately. The SVG objects from the Scan 'N Cut software were exported to the Brother software, which created the files for the papercrafting machine itself, so that the individual pieces could be drawn and cut.

The construction began by gluing the yellow circle to the substrate, a black foam board. The outlines for the starfish were drawn on the yellow circle. This allowed the starfish to be glued accurately on the yellow paper. The complete network of blue starfish was glued down next. Since the starfish form all the edges of all the shells, the positioning of the shells is obvious once the starfish are all in position. However, the shells were not glued to the paper, but were attached with thin pieces of double sticky foam tape. This set the shells off the yellow paper by about 3 mm, allowing interesting effects with shadows, as shown in Figure 10.



Figure 9: *The Brother ScanNCut SDX225.*



Figure 10: *An oblique view of the 3D effect.*

In designing the papercrafted shell pattern, there were three aesthetic considerations that we took into account; the colors of the shells, the texture of the paper, and a way to achieve a partial 3D effect. Since Shier had photographed undersea creatures, she chose a red for the scallops like that of the sunset scallop shown in Figure 11. Similarly she chose a blue for the starfish like that of the Micronesian blue sea star shown in Figure 12 — quite different than the pale yellow that Escher used in Regular Division Drawing 42. The



Figure 11: *A Sunset Scallop.*

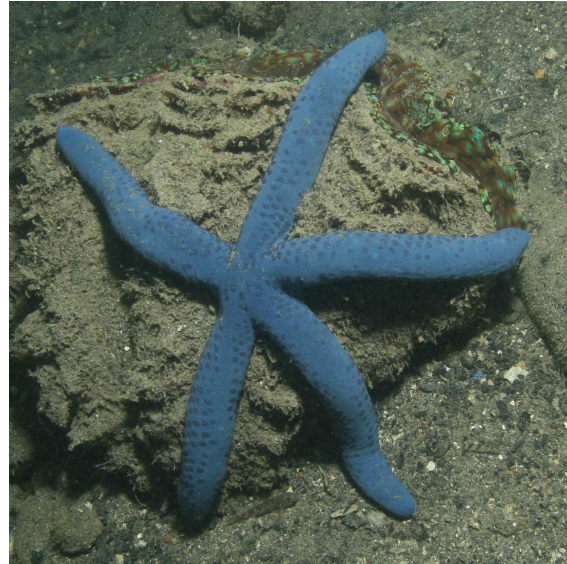


Figure 12: *A Micronesian Blue Sea Star.*

conchs are pale green as in Drawing 42, but the snails are pink rather than the brown of Drawing 42.

We rejected two choices for the texture of the paper: solid color which was not very interesting, and patterned paper which was a problem since the shells vary in size but the pattern doesn't. The solution was to use watercolor paper which has subtle features at all scales. This choice limited the size of the paper available to 8.5"×11", as mentioned above.

We also wanted to achieve a partial 3D effect in order to avoid an otherwise flat work with all shells pasted at the same level. We decided to raise the shells and snails slightly above the background starfish by cutting and attaching double sticky foam tape to them in the same shapes, as mentioned above. We did this because the "background" starfish touch each other and separate the scallops, conchs, and snails. This effect, shown in Figure 10, creates an interesting play of light and shadow not seen in Escher's Drawing 42 or the pattern of Figure 5. Figure 1 shows the final artwork.

The Free-Standing Lace Dodecahedron

We are interested in using free-standing lace to make polyhedra. Our first goal in this quest was to make a dodecahedron. In order to give it a more mathematical flavor, we decorated each face with a Greek letter that had mathematical significance. The open lace allows the viewer to partly see through into the interior of the dodecahedron. We used the Husqvarna Epic embroidery machine that we used for the embroidered hyperbolic fish pattern last year [2], though a lower-end machine would have sufficed.

We also wanted to color the 12 faces of the dodecahedron in a symmetric way. We used six colors from the color wheel in this order: red, orange, yellow, green, blue, and purple. On the top six pentagons of the dodecahedron, if the Greek letter was one color, its background was the next color in the color wheel order, so

that we had the (letter color, background color) pairs: (red, orange), (orange, yellow), (yellow, green), (green, blue), (blue, purple), and (purple, red). On the bottom six pentagons, the background color was the previous color in color wheel order, so we had the same pairs of colors, but in reverse order: (orange, red), (yellow, orange), (green, yellow), etc. Figure 2 shows the “bottom” pentagon (green, yellow) facing the viewer; the other “bottom” pentagons surround it.

However, to gain experience in making a lace dodecahedron, we decided to try something simpler first: a free-standing lace cube. To this end, we stitched a floral pattern using variegated thread on the square lace background. We used each of the six colors mentioned above for each of the square faces.

Figure 13 shows the computer controlled Husqvarna Epic finishing the sewing of the yellow square on a stabilizer sheet (to keep the stitches in place). Figure 14 shows the warped blue square after the stabilizer had been dissolved. It turned out that the warping problem could be fixed by ironing with a pressing cloth. Figure 15 shows the assembled cube.

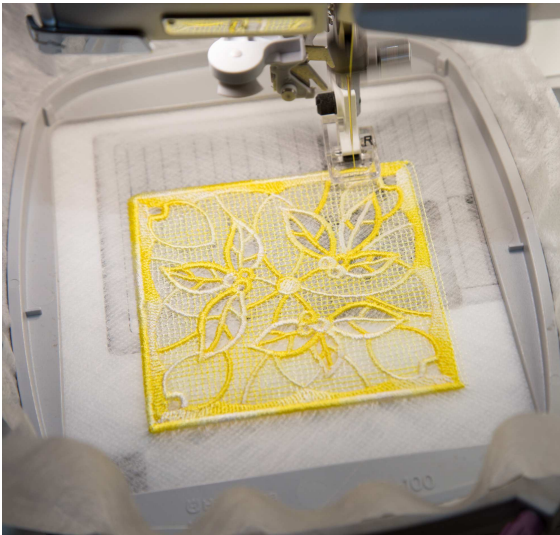


Figure 13: *The Epic sewing a Yellow Square.*

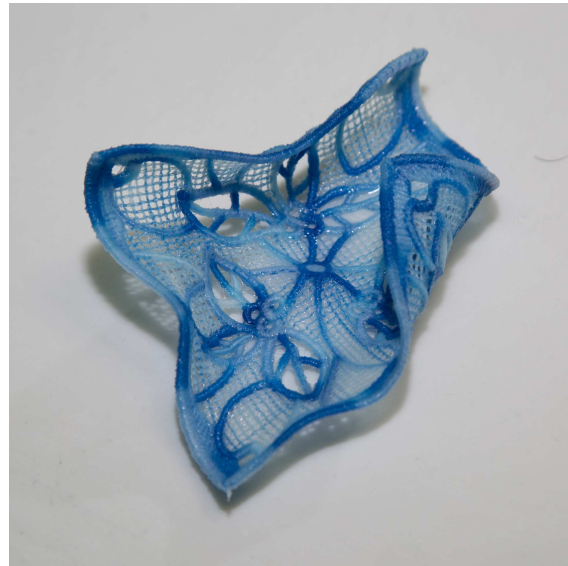


Figure 14: *The Warped Blue Square.*

The construction of the dodecahedron started with the design of the lace panels forming the faces. Those faces were designed with the Floriani Total Control-U professional level embroidery software. The newer versions of this software have support for the creation of free-standing lace. No suitable font was found for the Greek letters, so the letters were individually designed by Shier using basic shapes. The final product from the Floriani software is a file known as an “embroidery design”. This is effectively a machine language format file for an embroidery sewing machine. The Husqvarna Epic sewing and embroidery machine was used to sew out the lace panels. As with the cube, a stabilizing substrate is required for the sew out process, to hold the thread in place as the lace is created. We used the Embellish Rise Away Mesh substrate. After the lace panel is sewn, it is put into water, in which the substrate dissolves, leaving only the thread. This leaves a warped panel, like that shown in Figure 14, which is then ironed flat. Figure 16 shows a pentagon decorated with the Greek letter Π in one of the color combinations we desired.

Figure 17 shows an arrangement of all 12 pentagon faces, somewhat in the way they would be assembled into the dodecahedron. It was possible to sew the pentagons together along their edges to form a dodecahedron. But the lace panels didn’t have exactly identical side lengths because the lace mesh is slightly distorted by



Figure 15: *The completed lace cube.*



Figure 16: *A colored Π pentagon.*

the letters sewn into it. For this reason, directly sewing the lace panels together resulted in a “deflated” dodecahedron, as shown in Figure 18. Consequently, we decided to build a stiff framework both to provide matching edges and to keep the lace flat. That framework was created from poster board using the Brother ScanNCut SDX225. The ScanNCut software was not required, as the basic Brother software sufficed to create simple pentagons. The lace panels were attached to the poster board pentagons with Elmer’s glue. The pentagons were formed into a dodecahedron by attaching the edges on the outside with sticky back vinyl that was cut to exactly the right length and shape with the ScanNCut. This material is very similar to electrical tape, but comes in 12"x12" sheets and can be cut into pieces of arbitrary complexity with papercrafting machines. The resulting dodecahedral artwork is shown in Figure 2.



Figure 17: *An arrangement of all 12 pentagons.*

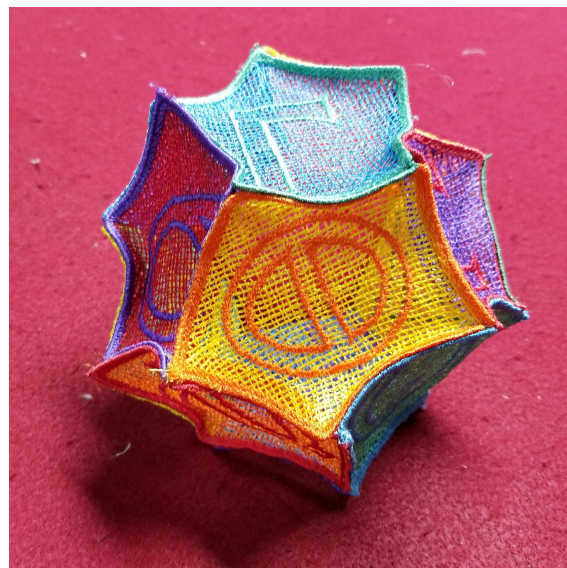


Figure 18: *A “deflated” lace dodecahedron.*

Educational Implications

Both the hyperbolic papercraft work and the machine embroidered dodecahedron were constructed using craft-scale computer aided manufacturing, and just as with 3D printing, are thus accessible to many educational institutions. The use of such machines can be entry points to learning about advanced manufacturing and robotics, both subjects of intense interest to students, parents, local governments, and potential new employers. This type of work may be of interest to students who are more tactile learners than visual learners, a group that traditional education sometimes has trouble reaching. Thus, we expect that projects like those presented here might appeal to some individuals who are not otherwise attracted to traditional mathematical instruction. And unlike last year's embroidered hyperbolic fish pattern, which used a top of the line embroidery machine, both of this year's projects can be created with relatively inexpensive machines (less than \$500 each), making them more accessible to educational institutions, makerspaces, and the general public.

Conclusions and Future Work

We have shown two mathematical art works that were constructed using craft technologies that are accessible to ordinary citizens and schools, and which could have educational benefits. One obvious direction of future work would be to papercraft other hyperbolic disk patterns. As an extension of the lace dodecahedron project, we have also considered making lace patterns on triply periodic polyhedra. Of course such patterned polyhedra could also be constructed by papercrafting.

As noted above, the (p,q) shell patterns exist not only in hyperbolic geometry, but also in spherical and Euclidean geometry. So one could papercraft shell patterns on each of the Platonic solids, corresponding to the regular spherical tessellations. One could also make free-standing lace or papercrafted shell patterns corresponding to the three regular Euclidean tessellations $\{3,6\}$, $\{4,4\}$, and $\{6,3\}$. Of course Escher has already created the seminal $(4,4)$ shell pattern in his Regular Division Drawing 42 based on the $\{4,4\}$ tessellation.

We could also consider a simpler pattern of shells inspired by Escher's confectionery box of Figure 4, and based on the regular tessellation $\{p,q\}$, in which there would be a group of p scallops centered on the faces and q -fold starfish at each vertex. We would probably want to restrict q to be 5 since most starfish have five legs.

Acknowledgements

We would like to thank the reviewers for their useful comments.

References

- [1] D. Dunham, J. Lindgren, and D. Witte Morris. "Creating repeating hyperbolic patterns." *SIGGRAPH Conference Proceedings*, 1981, pp. 215–223.
- [2] D. Dunham and L. Shier. "Embroidery of a Hyperbolic Fish Pattern." *Bridges Conference Proceedings*, Linz, Austria Aug. 16–20, 2019, pp. 211–216.
<http://archive.bridgesmathart.org/2019/bridges2019-211.html>.
- [3] M. C. Escher official website on symmetry patterns.
[https://mcescher.com/gallery/symmetry/#iLightbox\[gallery_image_1\]/35](https://mcescher.com/gallery/symmetry/#iLightbox[gallery_image_1]/35) (accessed Jan. 30, 2020).
- [4] C. MacGillavry. *Fantasy and Symmetry: The Periodic Drawings of M.C. Escher*. Harry N. Abrams, 1976.
- [5] D. Schattschneider. *M.C. Escher: Visions of Symmetry* New Edition. Harry N. Abrams, 2004.