Dancing the Quaternions

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Abstract

Swirling movements, popular among contemporary dancers and choreographers, often embody double rotations of the limbs similar to the Dirac plate or belt trick, in which an object attached to a stationary body by a flexible cord returns to its original state after a rotation of not 360° but 720°. This phenomenon is also seen in the Balinese candle dance, baton twirling, poi, and other performance forms. It is efficiently modeled by the quaternions and illustrates the mathematical theorem that the group SU(2) double covers the rotation group SO(3). We will look at how this plays out in dance and other performing arts, give some suggestions for simple and enjoyable movement tasks that illustrate the concepts, and see how comprehending the embodiment of the quaternions helps us better understand both the mathematics and the relevant movement arts.

Introduction

This paper is inspired by the author's fascination as performer, choreographer, and audience member with swirling movements in dance. It is also an attempt to better understand movements with the simple prop of a piece of paper in a series of dances and workshops created by the author and Erik Stern in the 1990s [12; 13; 14].

We start with a simple gimmick with the arms that has surprising connections to classical dance forms such as ballet, but also important connections to robotics. James Tanton has popularized what he calls the International Math Salute, a humorous trick involving twisting of the arms [19]. Here is a variation on another simple set of arm movements the author first learned from John Conway, but which is also a bit of mathematical and movement folklore; for example see the reference to a similar sequence related to the martial art of Wing Chung in Louis Kauffman's book *Knots and Physics* [7, pg 436].

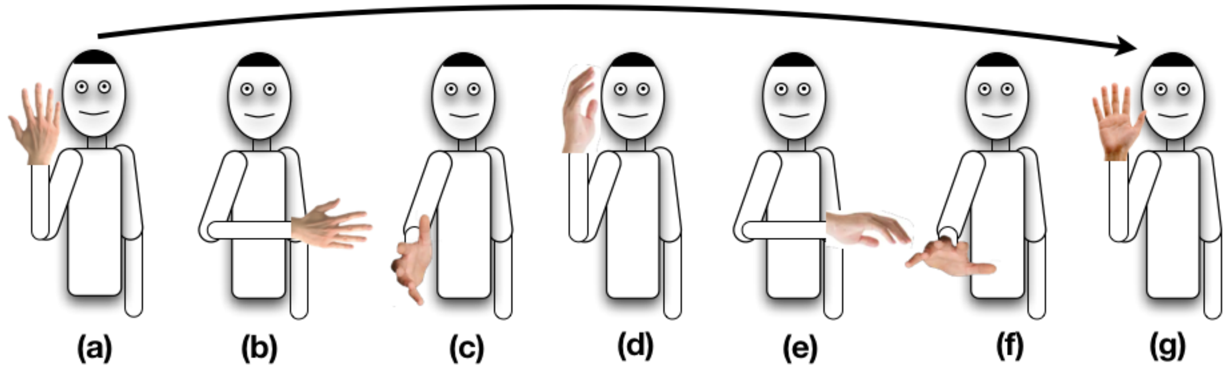


Figure 1: *The "Mathematical Hello," composed of six 90° rotations around three mutually perpendicular axes.*

We might call this version, shown in Figure 1, a "Mathematical Hello". Begin with the right arm held vertically in front of the body with the palm facing you and the upper arm perpendicular to the body, as in Figure 1(a). By simply twisting the right forearm 180° clockwise (as seen from above), one arrives at the position in Figure 1(g). But is it possible to get to that position from (a) without twisting the forearm? This may be accomplished with a series of six 90° rotations. Begin by rotating the upper arm 90° to the left so the hand is pointing to the left and the palm faces you, as in (b). Then rotate the forearm 90° to the front arriving at position (c), and then 90° up to position (d). At this point notice that the forearm has twisted 90° of the required 180°. Repeat the same three rotations to get to positions (e), (f), and (g). Finally wiggle the fingers "hello!" to complete the Mathematical Hello.

Here is a perhaps naïve explanation for how this works. Unflex the arm at the elbow so you start with the palm facing up. Then flex at the elbow 90° upwards, rotate the upper arm 90° left, and then unflex at the elbow 90° to the front, leaving the palm now rotated 90° to the left. The last elbow flex and unflex have "undone" themselves! Repeating these steps leaves the palm down as the upper arm has rotated a total of 180°. Now flex the arm at the elbow back to its original starting position and the palm will face front.

How might this be related to dance? Figure 2 shows arm arrangements similar to ballet or modern dance second, first, and fifth positions but here simplified - the actual arm positions are rounded with palms more closely facing (a) each other, (b) the torso, and (c) the forehead. A common sequence might be to move the arms directly from second (a) to first (b) and then to fifth (c) position. Note that if one were then to move from fifth back to second, a 90° twist or rotation of each arm would need to occur so as to bring the palms to their second position facing, a maneuver familiar to all dancers executing these exercises. The sequence from second to first to fifth, like those in Figure 1, utilizes two 90° rotations for each arm around perpendicular axes, however, this time with both rotations occurring via the shoulder joint.

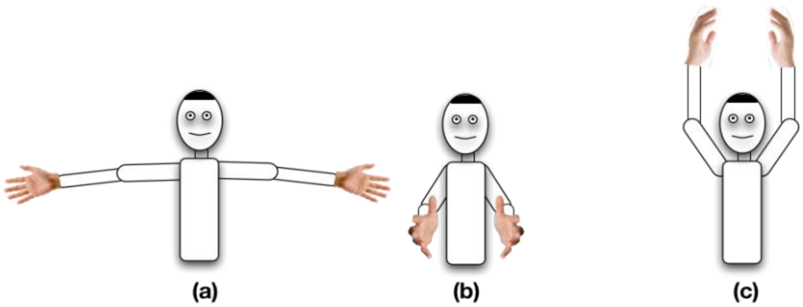


Figure 2: *Ballet and modern dance arm positions, modified: (a) second, (b) first, and (c) fifth.*

The repetitive exercise of such sequences of arm positions, known as port de bras, or "carriage of the arms" are central to ballet and associated dance forms. For example, in a recent 90-minute ballet class, taught by Rebecca Blair [1], whose classes I have taken for many years, I counted approximately 380 port de bras movements, or about one every 15 seconds. Port de bras are central to proper alignment, mechanics, form, and expression. Interestingly, the twists or rotations of the arms necessary to adjust the move from one established arm position to another described above seem to be rarely codified or even mentioned in dance literature and vary quite a bit – for example see the popular instructional video [11].

The dance theorist Rudolf Laban and his followers such as Irmgard Bartenieff, the developer of Effort /Shape and Bartenieff Fundamentals [20], often positioned the moving body within imagined frameworks of Platonic solids and extended the limbs towards vertices of these polyhedra in sequences known as movement scales. However, they also do not seem to have specified the twist or rotation of the limbs within these sequences. Gregg Lizenbery, long-time Director and Graduate Chair of Dance at the Univ. of Hawaii at Manoa, and international expert in Bartenieff Fundamentals and Laban Movement Analysis agrees, and points out that, "The choice to rotate the arms gradually or all at once when moving from one arm position to another varies from dancer to dancer and teacher to teacher." He says that he has also not seen clarifications of when or whether to rotate the arms during the Laban movement scales [9].

We may clarify aspects of arm rotations by referring to Euler's rotation theorem [21], which states that in three-dimensional space, any displacement of a rigid body such that a point on the rigid body remains fixed, is equivalent to a single rotation about some axis that runs through the fixed point. In the case of standard port de bras movements the fixed point is the shoulder joint. The overall rotation produced by the composition of two 90° rotations about perpendicular axes might be most easily visualized by using what we might humorously call the 1 by 1 by 1 Rubik's cube, illustrated in Figure 3. Here and in the rest of the paper the x-axis is directed outward toward the reader, the y-axis to the reader's right, and the z-axis upwards. The two 90° rotations shown are the two in Figure 2 for the right arm from second to first and then to fifth position. To bring the right arm back to second position would require a single rotation of 120°, not 90°, about the long diagonal shown in Figure 4(e), that rotation creating a cycle of the three vertices shown by dots. The correlation with the port de bras shown in Figure 2 is that the right hand is coincident with the letter A, at the center of the left face in (b), the front face in (c), and the top face in (d), with the shoulder joint at the center of the cube.

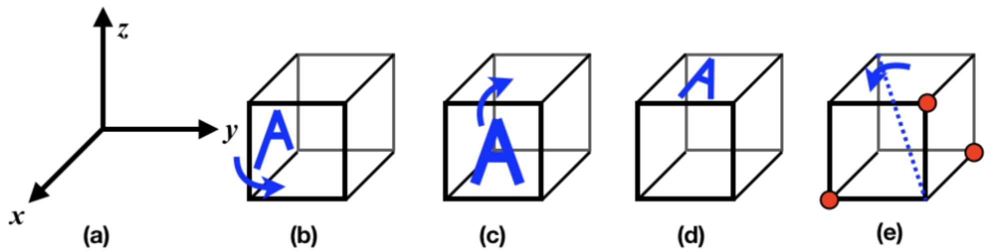


Figure 3: *The composition of 90° rotations about the z-axis (b) to (c), and the y-axis (c) to (d), is equivalent to a 120° rotation about the long diagonal shown in (e).*

The 120° rotation noted is certainly contrary to the seeming intuition that the arm must be twisted in the socket by 90°. Here is how dancers typically accomplish the movement: twist the arm 90° at the top in fifth position followed by a 90° rotation down to second position; or rotate the arm to second position followed by the 90° rotation of the arm in the socket while the arm is to the side; or gradually rotate in the socket as the arm moves from fifth to second position. However, a single rotation by 120° would yet feel different, as it would be 1/3 of a circle around the long diagonal in Figure 3(e). All three of the 90° motions are commonly used by dancers, even though again, at least in the ballet and modern dance idiom, they are often not discussed or clarified. This lack of specificity is not the case for all dance forms; for example, the author has performed Bharatanatyam, a classical dance form of southern India, which involves complex twisting motions of the arms that are carefully specified.

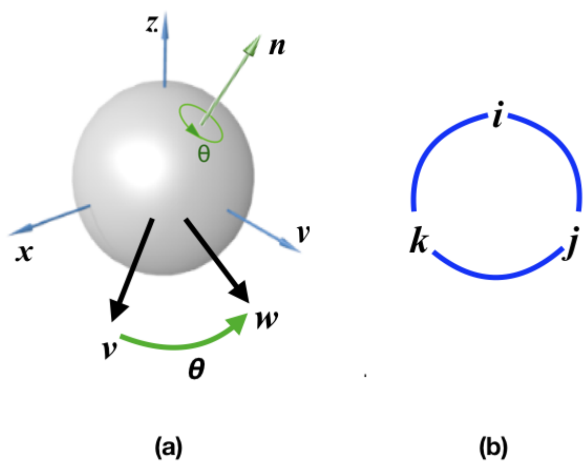
In the ballet idiom many of the transitions between arm positions approximate right-angle movements around the three axes used in Fig. 3.These right angles generate the complete set of 24 rotational symmetries of the cube, S_4. However, we will see that these are not sufficient to approximate all arm movements.

Of course, more complex motions of the arms than those above are not only possible but used constantly in nearly every dance form. The arms have at least five degrees of freedom allowing complex rotations: the shoulder ball and socket joint, the wrist's ellipsoidal joint, elbow and finger hinge joints, and the three associated pectoral girdle joints connecting scapula, ribs, clavicle and the sternum. Within the elbow joint additional motion is allowed by the radius's ability to partially rotate around the ulna. A coordination that we take for granted, such as reaching for the butter and passing it to someone else, requires quite complex sequences of joint motions. In the field of robotics, repeated sequences of the Mathematical Hello movements, for example, might cause a robot's arm to break off if the accumulated rotations or twists of the arm are not compensated for! Similarly, 3-D character animation, computer graphics, and virtual reality applications [5] commonly employ the quaternion mathematics outlined in this paper.

Quaternion Representation

In order to address the swirling movements that are the primary subject of this paper there are several popular mathematical formulations for handling three-dimensional rotations: matrices, Euler angles, axis angle, and quaternions. We will focus on using the quaternions combined with the axis angle approach. For detailed background on the mathematics, which will only be cursorily described here, see [4] or [8].

Figure 4(a) shows the rotation axis $\boldsymbol{n}$, and angle of rotation around $\boldsymbol{n}$ by angle θ which carries vector $\boldsymbol{v}$ in the plane perpendicular to $\boldsymbol{n}$ to vector $\boldsymbol{w}$. Briefly, quaternions are a 4-dimensional non-commutative division algebra $\mathbf{H}$ with elements of the form $\boldsymbol{q} = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} = a + \boldsymbol{v}$, where a, b, c, and d are real numbers, $\boldsymbol{v}$ is called a pure vector, $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ satisfy the multiplication rules $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$, with norm $\|\boldsymbol{q}\| = \sqrt{a^2 + b^2 + c^2 + d^2}$. Also, $\boldsymbol{ij} = \boldsymbol{k}$, $\boldsymbol{jk} = \boldsymbol{i}$, $\boldsymbol{ki} = \boldsymbol{j}$, with reverse multiplication order giving negatives, $\boldsymbol{ji} = -\boldsymbol{k}$, $\boldsymbol{kj} = -\boldsymbol{i}$, $\boldsymbol{ik} = -\boldsymbol{j}$, indicated by the circular formation in Figure 4(b). The elements $\pm\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ from **H** form the multiplicative group Q_8 shown in Figure 4(c). The unit norm quaternions $|\boldsymbol{q}| = 1$ form the three-dimensional sphere S_3 in $\mathbf{R}_4$.



	1	−1	*i*	−*i*	*j*	−*j*	*k*	−*k*
1	1	−1	*i*	−*i*	*j*	−*j*	*k*	−*k*
−1	−1	1	−*i*	*i*	−*j*	*j*	−*k*	−*k*
i	*i*	−*i*	−1	1	*k*	−*k*	*j*	−*j*
−*i*	−*i*	*i*	1	−1	−*k*	−*k*	−*j*	*j*
j	*j*	−*j*	*k*	−*k*	−1	1	*i*	−*i*
−*j*	−*j*	*j*	−*k*	−*k*	1	−1	−*i*	*i*
k	*k*	−*k*	*j*	−*j*	*i*	−*i*	−1	1
−*k*	−*k*	−*k*	−*j*	*j*	−*i*	*i*	1	−1

(c)

Figure 4: *(a) Axis-angle notation for rotations. (b) Circle for multiplications of* ***i***, ***j***, *and* ***k***. *(c) Multiplication table for quaternion group Q_8.*

The quaternion for a rotation by angle θ around pure unit vector n, as in Fig. 4(a), for $0 \le \theta < 4\pi$, is

$$\boldsymbol{q} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\boldsymbol{n}$$

The appearance here of the half angle $\frac{\theta}{2}$ may be surprising, especially since in the complex plane a rotation by angle θ around the origin is produced by multiplication by the complex number $\cos(\theta) + i\,sin(\theta)$. However, the half-angle is necessary, and we will see that it corresponds directly to rotational movements of the arms. Here $\boldsymbol{q}$ and $-\boldsymbol{q}$ generate the same three-dimensional rotation, accounting for what is called the "double-covering" of the set of three-dimensional rotations, the special orthogonal group SO(3), by the set of unit norm quaternions, the special unitary group SU(2). There are two types of calculations that are of interest to us. A rotation $\boldsymbol{q}$ followed by a rotation $\boldsymbol{r}$, is given by the total rotation $\boldsymbol{rq}$, where the first rotation is on the right in the multiplication, as it would be if we used matrices to represent the rotations. And the result of rotating the pure vector $\boldsymbol{v}$ around the pure vector $\boldsymbol{n}$ by angle θ using quaternion $\boldsymbol{q}$, as in Figure 5(a), is given by the calculation $\boldsymbol{w} = \boldsymbol{qvq}^{-1}$. The inverse of the quaternion $\boldsymbol{q} = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$ is $\boldsymbol{q}_{-1} = \frac{\boldsymbol{q}^*}{\|\boldsymbol{q}\|^2}$ where $\boldsymbol{q}_*$ is the conjugate $\boldsymbol{q}_* = a - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{k}$.

For example, the quaternion calculations describing the motion of the right arm from second position to first and then to fifth position, as described above using the rotations of the cube, are as follows. The first 90° rotation around the z-axis is represented by $\boldsymbol{q} = \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\boldsymbol{k}$, and the second –90° rotation around the y-axis is represented by $\boldsymbol{r} = \cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right)\boldsymbol{j}$. Note that this is a –90° rotation since it moves upwards from the x-axis to the z-axis; we adopt the convention that positive rotations about the x-, y- and z- axes are those that respectively move directly 90° from y- to z-, z- to x-, and x- to y-axes. The total rotation is thus

$$\boldsymbol{rq} = \frac{1}{\sqrt{2}}(1 - \boldsymbol{j}) \cdot \frac{1}{\sqrt{2}}(1 + \boldsymbol{k}) =$$

$$\frac{1}{2}(1 - \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{-\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}}{\sqrt{3}}\right) = \cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)\left(\frac{-\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}}{\sqrt{3}}\right)$$

This is again a 120° rotation around the same axis, namely $-\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, that we found before. On the other hand, since the right arm begins at position $-\boldsymbol{j}$, opposite the positive y-axis, the calculation for transforming this vector results in $(\boldsymbol{rq})(-\boldsymbol{j})(\boldsymbol{rq})^{-1} = \boldsymbol{k}$, which is where the arm ends up. However, this is not that helpful since it does not show the amount of rotation or twisting of the arm!

The utility of the quaternion representation of rotations is better seen in movement sequences in which an arm is continuously rotated through a "double rotation" of 720° rather than just 360° before returning to its starting point. The fact that this returns the arm to its equivalent starting position is often referred to as the Dirac belt or plate trick, after the physicist P.A.M. Dirac, who used it to illustrate similar properties of the spin of elementary particles [3]. A "hand-waving" kinesthetic proof of this equivalence is found in [10]. See the top of Figure 5, in which the left-hand circles in front and then in back of the body with the palm always facing the direction of motion. Each 180° rotation may be represented by a multiplication by $\boldsymbol{i} = \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\boldsymbol{i}$, and the sequence does not resolve until arriving at $\boldsymbol{i}^4 = 1$. At $\boldsymbol{i}^2 = -1$ the hand is in its initial position, but the arm is twisted 360° from its initial orientation. The bottom of Figure 5 shows a double rotation of both arms at the same time, each arm 180° rotated from each other. The right hand in the diagram may maintain its palm always facing the direction of motion, while the back of the left hand's palm faces the direction of its motion. Such swirling circular movements with both arms are often used in dance, for example see brief clips of dances by the author [17].

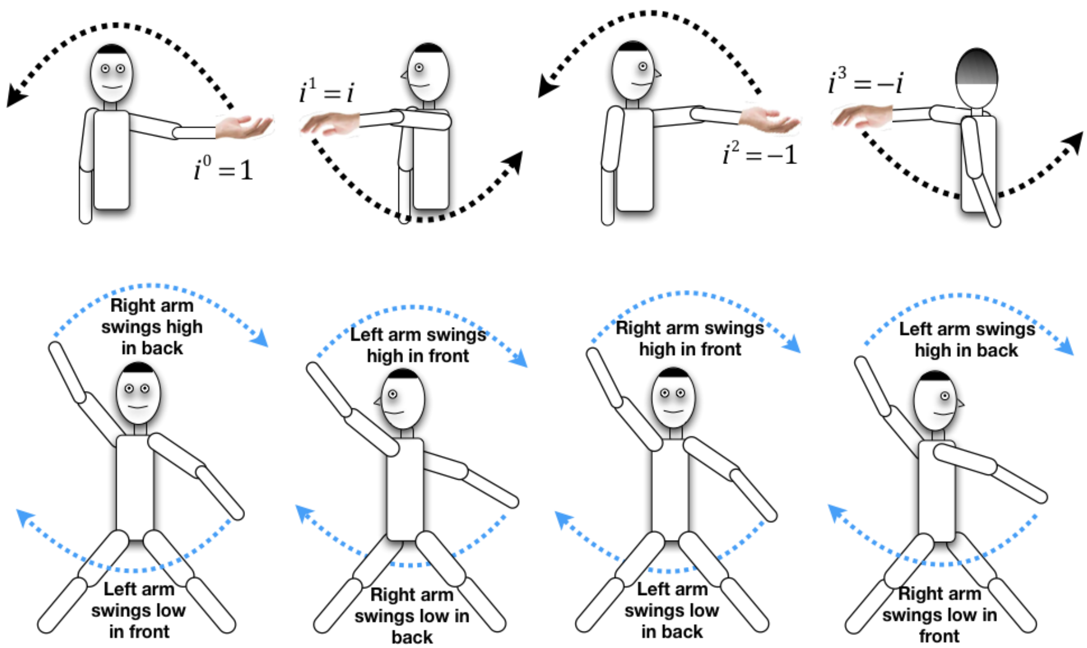


Figure 5: *Top: 720° rotation of the arm with palm facing direction of motion, represented by powers of quaternion* ***i****. Bottom: double rotation of both arms, 180° apart from each other.*

Such circular movements in which the palm faces the direction of motion were found to be helpful in a series of dances created by Erik Stern and I in the 1990s. In "Bartleby," 1991 [12], based on Melville's short story about a paper document copyist who decides "I would prefer not to" work anymore, we puzzled over a way to organize movement based on a character who no longer want to do anything – and who was assigned to work with mounds of paper. We began exploring ways for dancers to move with simple 8-1/2 by 11 sheets of paper, and eventually created an entire concert of dances [13;14] around manipulations of paper as dance prop. We found that movements with the paper work best if fairly constant and continuous pressure is maintained through the hand on the paper, by using the palm to smoothly push the paper.

Several short clips demonstrate some of these manipulations: [12] from "Bartleby" shows paper passes between dancers; the dance "Wind Tunnel" shows a variety of movements in which arm rotations and movements maintain constant pressure and motion against the paper – at one point instead of rotating the arm with the paper we instead rotate the body with a back roll while keeping the paper stable [14, 0:08 and 0:35]. These dances also led Stern and I to create workshops in which dancers play with and create movement sequences using ordinary paper. Workshop handouts are shown in Fig. 7 and also [18], and the reader may enjoy experimenting with these movements. Note that the 'figure-eight" movement with a sheet of paper in Fig. 7 might be understood as a slightly abrogated double rotation in which the rotations in two parallel planes are morphed into the same plane. Figure-eight movements, using that title, are popularly used and referred to in many forms of dance, and involve double rotations often in the same plane.

Double rotations of the arms or props are also common in many movement forms, for example poi, the Balinese candle dance, and baton twirling. In [17, 0:12-0:38] from the author's show *Twirl*, 2006, our long-time company dancer Saki manipulates a rope poi fashion using double rotations. In [2] the Filipino Cultural Dance Troupe uses double rotations of one arm at a time and also uses both arms in mirror image fashion. It is not difficult to use both arms candle dance style moving in double rotation fashion in a total of eight ways: in the same or mirror image orientations, either in phase or 360° out of phase, forwards or in reverse – and even two more ways in the same direction 180° out of phase, see [15].

In [14, 0:00-0:06] from the dance "Shadowed Flight" we see a clip in which we used double rotations with PVC pipe sections. In [17, 0:00-0:07] from "Interlude" we see a double rotation movement of an arm above the head which seems to help motivate a turn and leap. Once we recognize these kinds of swirling movements of the arms, we see them constantly in dance phrasing.

A Quaternion Dance Floor Pattern

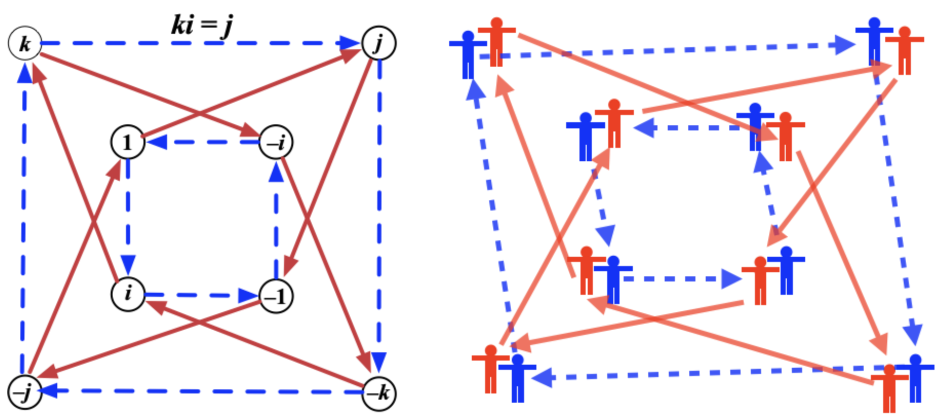


Figure 6: *(a) The Cayley color graph for Q8 generated by* ***i*** *and* ***j*** *with dashed arrows indicating multiplication by* ***i*** *on the right, and solid arrows indicating multiplication by* ***j*** *on the right. (b) shows each vertex replaced by dancing couples.*

In 2014 Vi Hart and Henry Segerman created a sculpture that is a 3-dimensional projection of a 4-dimensional figure that has the symmetries of the group Q_8 [6]. However, we have focused here on the use of the quaternions to describe movement sequences that proceed in time rather than existing at once in space. Here is a final suggestion for a danced form of the quaternion group Q_8. Figure 6 shows the Cayley color graph for Q_8 in which dashed arrows represent multiplication of the starting vertex by $\boldsymbol{i}$ on the right and solid arrows multiplication by $\boldsymbol{j}$ on the right. In the animation [16] we see the Cayley graph vertices replaced by dancing couples, with the dashed blue arrows representing movements of the blue dancers and solid red arrows movements of the red dancers. Though the author knows of no dance in which these sequences are used, I suspect there may be couple folk dances or ballet partner dances which approximate them, and this might also make for an interesting partner dance sequence.

Paper Dance and Math Workshop Ideas

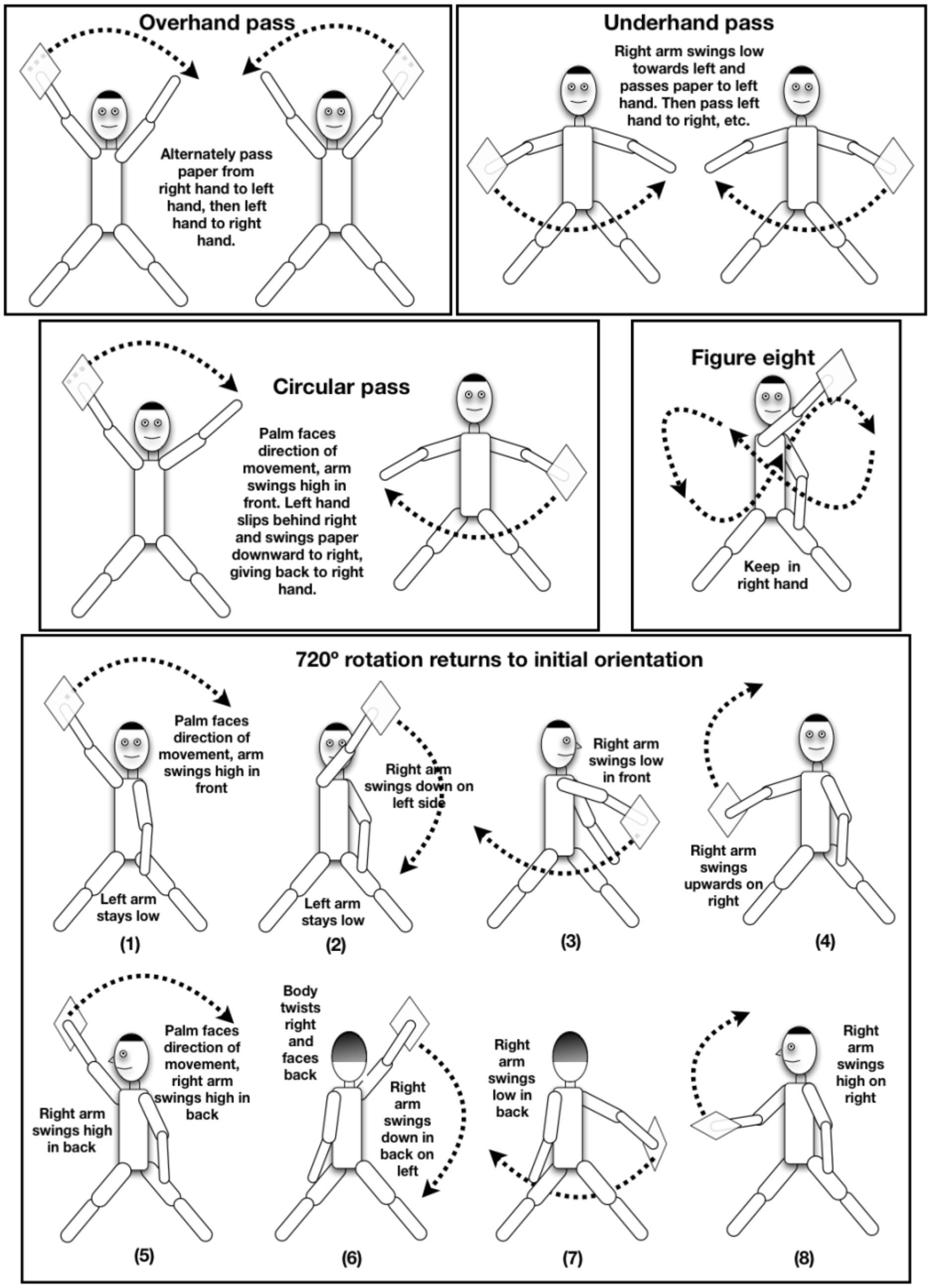


Figure 7: *Paper workshop handout.*

Methods of using paper as a dance prop, Figure 7, were developed by the author and Erik Stern in performances in the 1990s, including "Bartleby" [12] and *Dances for the Mind's Eye*, 1993 [13; 14]:

- Try using ordinary 8-1/2 inch by 11-inch sheets of paper.
- Dampening hands slightly with water or "tacky finger" helps – but work on not relying on aids.
- Try the simple tasks shown in Fig. 7. Create your own methods of moving with paper.
- Find ways to pass a single sheet of paper from one person to another.
- Try passing a sheet of paper around your back from one hand to another.
- Keep the palm of your hand moving constantly and steadily against the paper.
- Try in dance and math classes! Have students create sequences and perform to music.

- The 720° rotation sequence shows use of the quaternions for SO(3) rotations ($i^2 = j^2 = k^2 = -1$, etc.)
- Perform various simple combinations of rotations by π and/or π/2 around coordinate axes (without dropping the paper!) and explain resulting orientation in terms of quaternions or a cube's rotations.

Summary and Conclusions

In this paper I have given examples of dance movements which use double rotations that are perhaps best described using the mathematical model of the quaternions. These movements often arise fluidly within swirling dance sequences in a a variety of dance forms. A crucial artistic point is that in none of the dances displayed here are these double rotation movements included in order to display the mathematical ideas, rather they are simply part of the choreographic palette with which the dances are composed; for this reason, the mathematical concepts may only be approximately apparent to the observer. Although these dance movements often involve rotations with the arms, circles using leg gestures are also possible, especially in a movement known as fouetté, though the joints in the leg prevent the amount of rotation possible with the arms. This paper is also an attempt to encourage dance artists to pay more attention to the quaternion-like twisting that naturally occurs with gestures of the arms. And we encourage dance and math artists to observe quaternionic elements when playing with the dance prop of a simple sheet of paper.

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