Everted Embeddings

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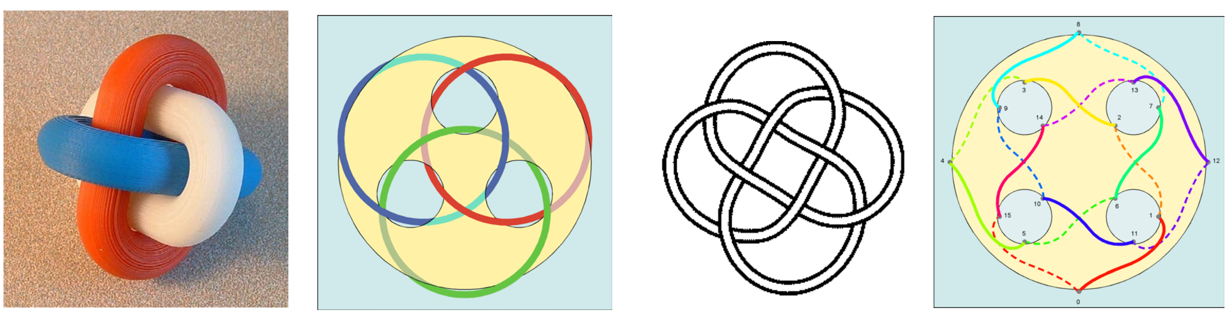
Abstract

Every mathematical knot or link can be embedded crossing-free in the 2-manifold surface of a handle-body of appropriate genus. This is an investigation of how such an embedding changes when the surface is everted. The cases studied concern Knot 8_{18} embedded in surfaces of genus 4 and the Borromean rings in surfaces of genus 3.

Introduction

In November 2018, Pedro Henrique Affonso, a Brazilian psychoanalyst, pointed me to a website [8] that raised some intriguing and challenging questions. This page first mentions the well-known fact that a TorusKnot(s,t), embedded in the surface of a torus, turns into a TorusKnot(t,s) when the torus is turned inside out. It then asks what would happen to the Borromean rings (Figure 1a) embedded in a 3-hole torus (Figure 1b), or to the Knot 8_{18} (Figure 1c) embedded in a surface of genus 4 (Figure 1d), if these surfaces were everted. In the following, I am answering the specific questions asked. I also present a new "hands-on" approach that can provide answers to similar questions, and which can build an intuitive understanding of the process of skin-everting a surface containing an embedded knot or link.



(a) (b) (c) (d)

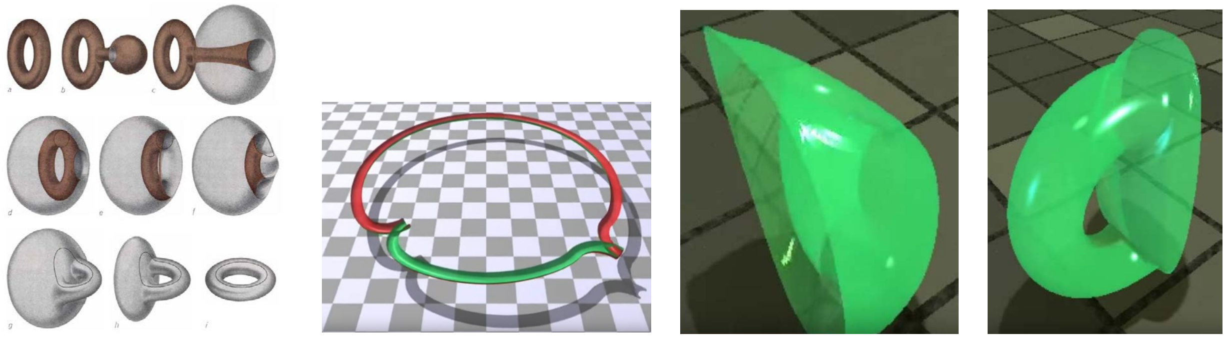
Figure 1: *(a) Borromean rings, and (b) their embedding in a 3-hole donut. (c) Knot 8_{18}, and (d) its embedding in a 4-hole torus.*

Different Surface Eversions

The question raised on this web page may be answered in several different ways. The key operation referred to is the process of *everting* a closed, smooth, orientable, non-self-intersecting 2-manifold, so that in the end the inside surface is now visible from the outside, while the original outer surface has become the inside one. But there is more than one way of doing this.

Computer Graphics can evert a surface by negating one of the three coordinates of the given model, e.g., multiply all the z-coordinates by -1. This creates a mirror image of the original 2-manifold, and it also flips all the surface normal vectors to their opposite directions. However, topologists prefer a smooth eversion process that never creates a "bad" 2-manifold with any tears or sharp creases. To get from the original shape to the mirrored handle-body, we follow every surface point described as (x, y, tz) as t changes from +1 to -1. When t=0, the whole 2-manifold is crushed into the x-y-plane, where it exhibits sharp bends and creases, and different segments of any knot or link strand will pass through one another.

Regular Homotopy is a process where the surface is allowed to pass through itself, without experiencing any cuts or tears, but maintains finite curvature everywhere at all time. Using such a process, it has been shown that spheres can be turned inside out [9]. The eversion of the torus has also been demonstrated by Phillips [5] (Figure 2a) and by Cheritat [1] (Figure 2b), and a Mathologer video [3] shows how this approach can also be applied to handle-bodies of higher genus.



(a) (b) (c) (d)

Figure 2: *Homotopic torus eversion by (a) Phillips, (b) Cheritat. (c, d) Skin eversion by McShane.*

Skin Eversion is yet another way to evert a surface. This one is closest to what can be done with a physical realization. A small puncture is placed at a convenient location in the surface, and then the whole surface is pulled through this pinhole. This must be done in such a way that no other parts of the surface ever pass through one another, and the whole surface stays smoothly connected everywhere except at the pinhole. For a sphere, this process can be visualized easily (think of a large inflatable beach ball). For a torus, this is harder to visualize; but McShane [4] has produced a good video to demonstrate this process (Figures 2c,d). The result is again a torus; but in this process, the lateral and longitudinal coordinate lines exchange roles. Thus, if we had drawn a small circular loop around the minor radius of the torus, after eversion we would find this circle circumnavigating the central tunnel of the everted torus.

Everting Embedded Knots and Links

Every mathematical knot or link can be embedded crossing-free in the 2-manifold surface of a *handle-body* of appropriate genus. (Just draw the knot on a sphere and add a handle to accommodate every drawn crossing.) A handle-body of genus *n* is a closed, orientable 2-manifold, topologically equivalent to a sphere with *n* simple handles that are neither linked nor knotted. Typically, one wants to find a handle-body of lowest genus that can provide the needed number of handles and/or tunnels to accommodate all the crossings that the given knot or link would exhibit when drawn into a paper plane. In addition, I like to find embeddings that are simple and elegant, and which offer a high degree of symmetry.

The main question I am addressing here is: What happens to an embedded knot or link when the host surface is everted? First some general observations: The regular homotopy process [1] is a smooth procedure to achieve "computer-graphic" mirroring; so this will produce a mirror image of the given knot or link. For knots that are not amphichiral, i.e., cannot be deformed into their mirror image, this will change the topology of the given knot. E.g., a left-handed trefoil would turn into a right-handed one.

For my investigation, I focus on the *skin-eversion* process, because this transformation will not change the topology of the knot and corresponds directly to the questions asked on the web page [8]. In this process, no portion of the surface ever passes through itself, except that everything passes once through the pinhole. This means that the knot or link cannot possibly change. On the other hand, one may now ask, how the observed embedding will change in this process. For mathematicians it may be sufficient to know that the eversion simply produces a different embedding of the surface in R^3, the Euclidean 3-space. All answers can then be obtained by moving the observer's point of view from the outside of the handle-body to its inside. This explains that handles turn into tunnels, and vice versa.

Therefore, a TorusKnot(s,t) will turn into TorusKnot(t,s). However, the initiator of website [8] wants to compare the initial knot and the everted version on the same type of handle-body. For people without sufficient computer-graphics skills to model this process virtually, I present a hands-on approach that involves several different physical models to represent various phases in the eversion process.

Physical Skin Eversions

To make sure that I realistically capture the skin eversion process without making any mistakes, I started to build tangible models from thin plastic bags that can actually be physically everted by pulling them through a relatively small opening in their surface.

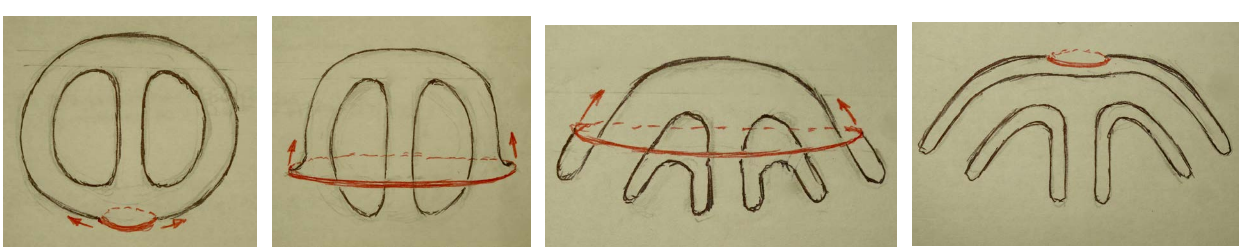
The pin-hole-eversion of a torus has been modeled on YouTube [4], and this process can readily be re-enacted with a plastic bag model. It turns out that the same basic approach can also be adapted for handle-bodies where more than two tubular, columnar *Pillars* emerge from a "South pole" and then merge again at the "North pole," as is the case in the *Stem* and *Pagoda* structures (Figures 5c and 5d).

A flexible model of a genus-2 handle-body approximating the shape of the bending-energy-minimizing Lawson surface [2] (Figure 3a) can be constructed by joining three cylindrical plastic sleeves in two 3-way junctions (Figure 3b). I cut a small, expansible hole near the South pole (Figure 3c) and then pull its expansible border upwards around the entire structure. In this process each *Pillar* gets transformed into a "reversed sleeve," forming a "mouth" like the one that can be found in the classical depiction of a Klein bottle. After the pinhole border has been pulled all the way to the North pole and has been sealed again, the resulting structure has the shape of a 3-way tubular *Junction* in which n=3 double-walled tubular stubs emerge from a central branch point (Figure 3d).

(a) (b) (c) (d) (e)

Figure 3: *(a) Genus-2 Lawson surface; (b) its bag model shaped as a 2-hole donut, or (c) with three symmetrical handles; (d) everted into a tubular 3-way Junction. (e) Poorly everted Tetrus surface.*

Figure 4 shows the deformations associated with this process in a cross sectional view. The perimeter of the expansible puncture is shown in red. It starts at the bottom (Figure 4a). As this border moves upwards around the whole structure (Figure 4b and 4c), the three *Pillars* turn into double-walled reversed sleeves. Eventually the process results in a clean tubular 3-way *Junction* (Figure 4d).



(a) (b) (c) (d)

Figure 4: *Cross sections of the skin-eversion of a genus-2 torus: (a) It starts with a small puncture, (b,c) which is pulled around the whole handle-body, and (d) results in a 3-way tubular Junction.*

The 3-way *Junction* can then readily be deformed into a thick spherical shell with 3 tunnels leading from the outside to the inner cavity. This represents a handle-body of genus 2 that is useful for analyzing what happens to any embedded knot or link, when that surface is skin-everted. Even though its geometry has changed considerably, it still has the 3-fold rotational symmetry of the original handle-body.

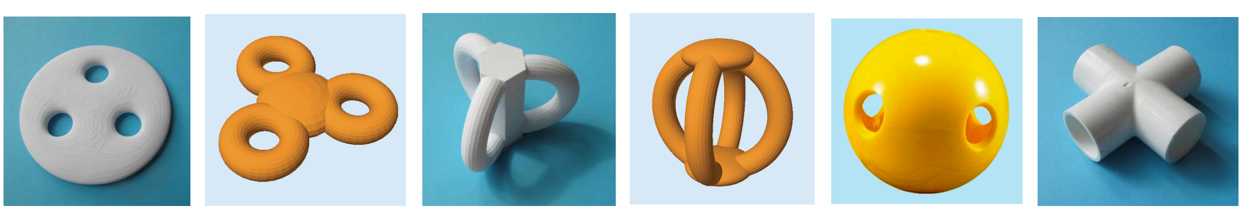
I made a couple of attempts to directly evert a plastic-bag model of a *Tetrus* – a "fattened" and nicely rounded model of a tetrahedral edge frame. If I placed the puncture in the middle of one of the six edge-handles, I obtained a "dog-bone" with four little stubby endings (Figure 3e); but this structure does not represent a simple handle-body useful for analyzing any embedded knots or links; it has too much internal structure with additional branching inside every "reversed sleeve" stub. These tangled surface regions cannot be fully everted and brought to the outside to form a simple handle-body, because the plastic bag material does not stretch enough.

It seems that nothing more complicated than handle-bodies with just parallel *pillars*, like the *Pagoda* or the *Stem* surfaces (Figures 3a, 5d; 5c), can be everted completely with a physical plastic-bag model. Mathematically, every handle-body surface can be skin-everted; but it would be much more challenging to make a computer-graphics movie of this transformation than the YouTube movie showing this operation for a torus [4]. The intermediate deformations of the surface would be much more extreme.

Symmetrical Handle-Body Shapes

I only know how to evert *n*-handle *Stem* shapes or (*n*+1)-pillar *Pagodas* into (*n*+1)-way tubular *Junctions*. Thus, if an eversion problem is presented on a different handle-body, I need to transform it into one of these special structures, which can then be everted as shown in Figures 3 and 4. To do this, I use the following symmetrical handle-body shapes of genus *n*:

- A genus-*n* torus in the shape of a *n*-hole *Donut* (Figure 5a).
- A slight change in geometry then leads to a *Disk* with *n* ring-shaped *Ears* attached (Figure 5b).
- A prismatic *Stem* with *n* handles attached to its sides. I call this an *n*-handle *Stem* (Figure 5c).
- A set of *n*+1 such handles, without the central *Stem*. For low genus, this resembles the energy-minimizing Lawson surface [2]. In general, it looks like a *Pagoda* with *n pillars* (Figure 5d).
- By focusing on the openings between the *pillars* of a *Pagoda*, this can be deformed into a thick-walled *Wiffle Ball* with *n*+1 tunnels to its interior around its equator (Figure 5e).
- This spherical shell can then be transformed into a tubular (*n*+1)-way *Junction* (Figure 5f).



(a) (b) (c) (d) (e) (f)

Figure 5: *Symmetrical handle-bodies of genus 3: (a) "3-hole Donut," (b) "3-ear Disk." (c) "3-handle Stem," (d) "4-pillar Pagoda," (e) "4-hole Sphere," (f) "4-way tubular Junction.*

In the following case studies I show how a given initial handle-body can be transformed into one of the special structures that I know how to skin-evert.

Case Study 1: Knot 8_{18} on Genus-4 Surfaces

Knot 8_{18} (Figure 1c) can be embedded nicely and symmetrically in a 4-hole donut (Figure 1d). I have given the knot trace an overall rainbow coloring to make it easier to follow the 16 trace segments in the right sequence. I want to maintain this 4-fold rotational symmetry in all my eversion studies. Thus, the most suitable evertible handle-body of genus 4 is not the *Pagoda* surface with five identical *pillars*, but rather a structure with a central *Stem* and four outer handles aligned with it. Figure 6 shows the construction of a corresponding plastic bag model.

I started with five cylindrical plastic sleeves (Figure 6a). First, I connected them to form a tubular 5-way junction (Figure 6b). Then I connected the other ends of the five sleeves to form a (red) *Stem* with four (lime-colored) handles (Figure 6c). At the upper end, I did not completely close up the region between the four lime handles to leave an opening through this plastic surface (Figure 6d). To do the physical eversion, I reach into this top opening and start pulling out the middle portion of the red stem; I also reach into the four handles and pull out their mid-sections (Figure 6e). The eversion is complete when the lower ends of all five sleeves have been pulled into the location of the final, double-walled 5-way *Junction* and the upper halves of all five sleeves have been everted and draped snuggly around their former lower halves (Figure 6f).

(a) (b) (c) (d) (e) (f)

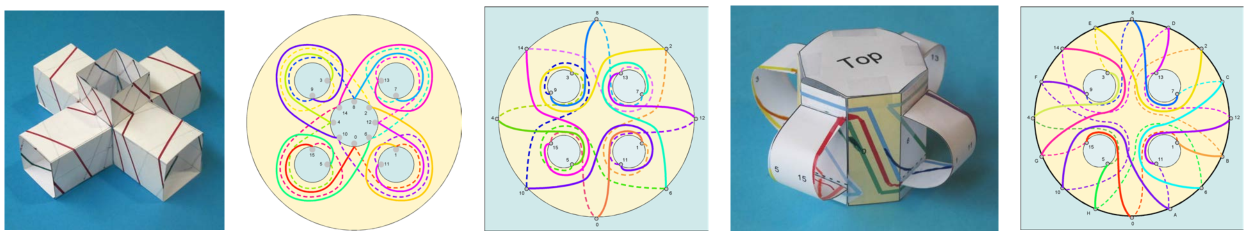
Figure 6: *Genus-4 plastic-bag model: (a) the parts, (b) one 5-way Junction, (c) 4-handle Stem, (d) showing opening for eversion, (e) partially everted, (f) completely everted.*

The next challenge I faced was how to embed Knot 8_{18} properly in the surface of this particular handle-body. In one approach, I started with the 4-hole *Donut* representation (Figure 1d) and pulled the four holes outwards to form a "*Disk* with 4 *Ears*" (Figure 7a). I then built a first simple paper model consisting of a central *Stem* in the form of an 8-sided prism, to which I attached four *Ear*-handles in the form of simple paper ribbons (Figure 7b). I initially placed the *Ears* in the common equatorial plane to mimic the original 4-hole donut; this makes it easier to transfer the given knot trace onto this paper model. I then turned the four *Ears* 90° CW (clock-wise), while keeping the trace topologically unaltered (Figure 7c). In a second paper model, I "fattened" the four handles by giving them square cross sections (Figure 7d). I then narrowed the *Stem* to a 4-sided prism with the same diameter as the handles (Figure 7e). Now the shape is close enough to my plastic-bag model (Figure 6d), so that I can transfer the knot trace correctly and draw it with some marker pens onto the bag model.

(a) (b) (c) (d) (e)

Figure 7: *Knot 8_{18}, (a) embedded on a Disk with 4 Ears, (b) and on a prism with 4 ribbon Ears. (c) Ears rotated 90° CW. (d) Handles with square profiles. (e) 4-sided prismatic Stem.*

After eversion, the model has the shape of a double-walled 5-way junction (Figure 6f). I read off the traces and copy them onto a simple paper model of a 5-way junction (Figure 8a). I then interpret this as a large spherical shell with five holes (Figure 8b). The opening of the *Stem* stub is identified with the central hole, and the ends of the four handle-stubs are identified with the four outer holes in the disk (Figure 8b). This structure in turn can be converted readily into a 4-hole *Donut* by doing a 2D circle inversion that enlarges the central hole and maps it to the perimeter of the disk (Figure 8c). Comparing this with the original embedding (Figure 1d), one can see that the trace now passes eight times through the outer horizon – corresponding to the eight passes of the trace across the mid-line of the stem (Figures 7c and 7d). The trace dives into each hole twice, but swirls around inside the hole through 1¾ turns.

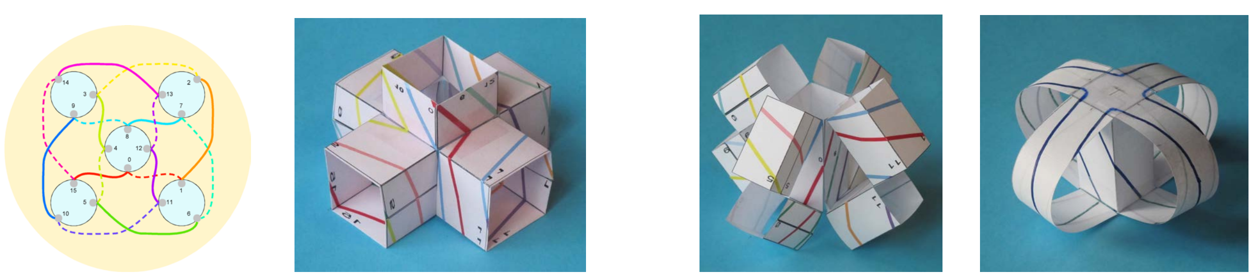


(a) (b) (c) (d) (e)

Figure 8: *(a) Knot 8_{18} on the everted 5-way Junction, (b) on a large spherical shell with 5 holes, and (c) on a 4-hole Donut. (d) New 4-handle Stem model; (e) new eversion result.*

I wanted to see, whether twisting the four *Ears* in the opposite direction would lead to a simpler result. I went through the sequence of model building once again with this different starting condition (Figure 8d). The resulting trace pattern looks quite different (Figure 8e): Two traces still pass through every hole, but no trace makes more than a full turn around the hole while passing through it. On the other hand, there are now 16 perimeter crossings. It is not clear which model is nicer or easier to understand.

I found it unsatisfactory that I had to decide which way to turn the *Ears* before applying the eversion process. Also, both everted embeddings are more complicated than I would have expected. Thus, I did a third eversion experiment, running the eversion process "backwards." I started with the simple 4-hole *Donut* embedding (Figure 1d), did a planar circle inversion of it (Figure 9a), and mapped the result onto a double-walled 5-way *Junction* model (Figure 9b). I then back-everted this model into a 4-handle *Stem* structure (Figure 9c) and compared this embedding (Figure 9d), to the two previous 4-handle *Stem* configurations (Figures 7c, 8d). The new embedding (Figure 9d) is definitely less complex. However, when I flipped the four handles of this model, either CW or CCW, to obtain corresponding 4-hole *Donut* models, I obtained the results shown in Figures 8c and 8e, respectively. This now shows that those two embeddings are essentially the same!



(a) (b) (c) (d)

Figure 9: *(a) Circle-inverted 4-hole embedding, to be mapped on 5-way Junction. (b) Genus-4 5-way Junction model. (c) Back-everted 4-handle Stem model. (d) Simplified paper model.*

Case Study 2: The Borromean Rings

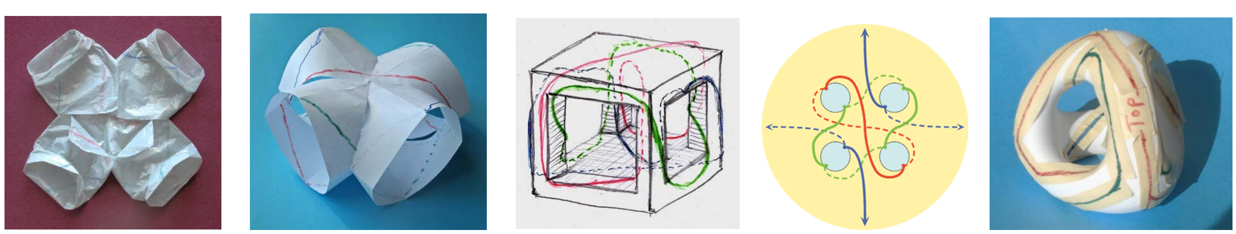
The second case study concerns the Borromean rings. The most symmetrical way to embed the three rings in a genus-3 surface is to use a *Tetrus* as the handle-body (Figure 10a). Since I don't know how to skin-evert this geometry, I transformed this embedding into a *Pagoda* structure. This can be done by spreading the top and bottom edges in Figure 10a into the squarish top and bottom plates seen in Figures 10b and 10c. Figure 10c shows a smooth 3D print model of this handle-body, with the three rings marked with colored tape. Figure 10d shows a corresponding plastic-bag model, which has been constructed in the spirit of Figure 3.

(a) (b) (c) (d) (e)

Figure 10: *Borromean rings embedded: (a) in a Tetrus, (b,c) in 4-pillar Pagoda structures, (d) in a corresponding plastic-bag model, and (e) in a large spherical shell with 4 tunnels.*

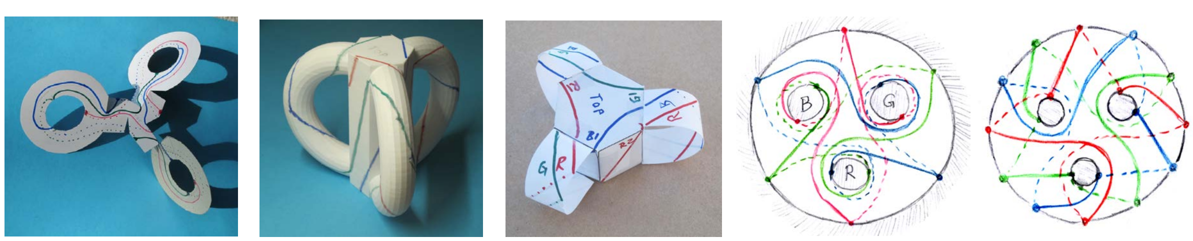
After physically everting the plastic-bag model (Figure 10d), resulting in a 4-way *Junction* shape (Figure 11a), I captured the geometry of the embedded traces in a simple paper model of such a 4-way *Junction* (Figure 11b). This intermediate visualization model made it easier for me to sketch the everted traces on a 4-pillar *Pagoda* (Figure 11c). This last configuration can also be depicted by showing the handle-body as a large spherical shell with four closely spaced holes (Figure 11d), which then can be compared easily to Figure 10e. This shows clearly, how the embedding differs from the original one. Now, three different strands wind around each of the four handles between two holes; and only two strands pass through each tunnel. Also, the red and blue lines cross with themselves at the top and bottom plates of the *Pagoda*, whereas originally they crossed each_other in those places.

To see this result on the original *Tetrus* handle-body, one can squish the squarish top and bottom plates of Figure 10c into *Tetrus* edges. Squishing the top plate front-to-back will create a top handle through which the red trace passes twice in the same direction (Figure 11e); whereas squishing it left-to-right makes the red trace pass itself in opposite directions. The bottom plate must be squished in the direction orthogonal to the top compression; the relative directions of the blue traces passing through this handle will then exhibit the same behavior as that of the red traces in the top handle.



(a) (b) (c) (d) (e)

Figure 11: *Everted Borromena rings shown on: (a) the plastic-bag model, (b) on a 4-way Junction, (c) on a 4-pillar Pagoda, (c) on a spherical shell with 4 holes, and (e,) on a Tetrus shape.*



(a) (b) (c) (d) (e)

Figure 12: *Borromean rings embedded: (a) on a 3-Ear disk, (b) on a 3-handle Stem 3D-print, and (c) on a simple paper model. Eversion results for (d) CW and (e) CCW Ear rotation.*

Unfortunately, in the above process, the original 3-fold rotational symmetry inherent to the Borromean rings has been lost. A slightly different process can maintain 3-fold symmetry. With this goal, I started with the embedding shown in Figure 1b, seeing this as a 3-*Ear Disk* (Figure 12a). This handle-body can be turned into a 3-handle *Stem* structure by following the discussions of Knot 8_{18}. Again, there is a choice whether to turn the *Ears* CW (Figure 12a) or CCW through 90°. For the first option, the resulting embedding is captured on a 3D print model (Figure 12b) and also on a simple paper model (Figure 12c). The application of the eversion process now follows the description in the previous section (Figures 7, 8). The everted embeddings for CW and CCW *Ear*-rotation are shown in Figures 12d and 12e, respectively.

As for Knot 8_{18}, I wondered whether the "back-eversion" process would result in a simpler embedding. I used a plastic-bag model of a 3-handle *Stem* (Figure 13a) and everted it to a double-walled tubular 4-way *Junction* (Figure 13b). This structure is very similar to a thick spherical shell with four tunnels leading to the inner space, as well as to a *Tetrus* structure. So I could readily embed three Borromean rings in the form of pipe-cleaners (Figure 13c) in the most symmetrical manner depicted in Figure 10a and draw the corresponding traces onto the plastic bags (Figure 13d).

(a) (b) (c) (d) (e)

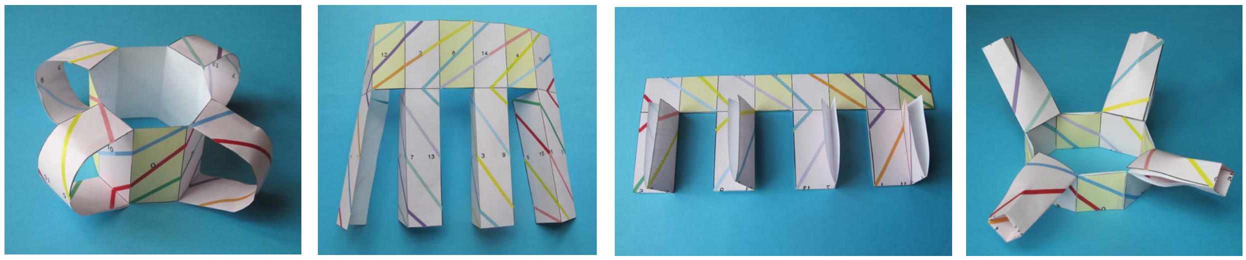
Figure 13: *(a) 3-handle Stem bag-model; (b) everted into a tetrahedral 4-way tubular Junction. Borromean rings: (c) inserted in the form of pipe-cleaners, and (d) drawn onto the bag model. (e) Everted surface shown as a simple 3-handle Stem model.*

I back-everted this model to the *Stem* configuration and I captured the result on a small paper model (Figure 13e). As for Knot 8_{18}, it looks surprisingly simple. All three rings now wind around every one of the three handles as well as around the *Stem* (initially all three rings passed through every tunnel). But now, only two rings dive through each of the three tunnels (initially two rings wound around each handle). To see the result on a 3-hole donut, I had to flip the *Ears* through ±90°. For a CW rotation, I obtained the same result shown in Figure 12d, and for a CCW rotation, it was the same as in Figure 12e.

Eversion Tracking with Paper Models

The plastic-bag models of any (n+1)-pillar *Pagoda* or n-handle *Stem* structure provide a robust, standard way of physically skin-everting a surface of genus n. Thus we can answer any given eversion question, as long as we can figure out how to transform the original handle-body into such a structure with (n+1) parallel *pillars*, or into a (n+1)-way tubular *Junction* model. If the original embedding is presented on an n-hole *Donut*, then deforming this handle-body into an n-*Ear Disk* and twisting all *Ears* through 90° will get it to resemble the desired *Stem* geometry. Alternatively, if we have to start from a *Wiffle-Ball* with n holes, it is conceptually easy to transform this into an n-way tubular *Junction*. In either case, we then have a handle-body shape that we know how to skin-evert.

Towards the end of my studies described above, I realized that I can construct directly a simple paper model showing the everted state on a tubular *Junction*. Since every handle or *pillar*, as well as a *Stem*, evert into double-walled tubular stubs, I can construct these "reversed sleeves" by simply folding each paper segment, used to form a *Stem* or a *Pillar*, back onto itself before closing it into a prismatic tube, and then glue these double-walled pieces together. Moreover, I can simply mirror the trace pattern used on the initial handle-body to obtain the proper everted traces on the folded-up parts that made the n-way *Junction* model. Figure 14 shows that this model construction works even for an "unbalanced" model with an 8-sided *Stem* and flat, two-sided handles. I can still read off the desired topological information of how many times a trace wraps around a handle or passes through a particular tubular stub.



(a) (b) (c) (d)

Figure 14: *(a) 4-handle Stem model; (b) the paper cut-out used to construct the Stem model; (c) the folded-up, mirrored cut-out; (d) the resulting tubular 5-way Junction model.*

Discussion

This work has also led to an exciting discovery. There is a close relationship between the Borromean rings and Knot 8_{18}. When looking at the corresponding *n*-hole *Donut* embeddings, we can see that the former is simply a 3-fold symmetrical version of the latter, which has 4-fold rotational symmetry. This is true before and after the eversion process. Indeed, I can actually take ¾ of any paper model of Knot 8_{18} (Figure 7c) and re-glue it into a structure with 3-fold rotational symmetry; then, after recoloring the traces, I have a copy of an equivalent Borromean paper model (Figure 12c).

What is the origin of this special, close relationship? The Borromean link as well as Knot 8_{18} can be seen as TorusKnot(3,3) and TorusKnot(4,3), respectively, that have been turned into alternating knots. This then leads naturally to the conjecture that other TorusKnots(*s*,3) would also lead to equivalent embeddings with *s*-fold rotational symmetry. For *s*=2, this leads to Knot 4_1, the Figure-8 Knot; and for *s*=5, this leads to Knot 10_{123} [6]. Additional experiments have shown that for these cases one obtains the equivalent versions of all the models described in the central part of this paper. All these types of knots are fully amphichiral, i.e., they can be deformed into their mirror images.

Converting an embedding on an *n*-hole *Donut* into an *n*-handle *Stem* offers some options. The *n* handles can be twisted through 90° in a CW or CCW manner. It is not immediately clear which one of the two options is preferable. The two options are not truly different embeddings, since they can be smoothly transformed into one another by deforming the handle-body, – specifically, by twisting the *Ears* through 180°. The two different-looking skin-everted embeddings (Figures 8c and 8e) or (Figures 12d and 12e) resulting from these two options are also essentially the same, respectively.

However, there is some freedom in how one wants to draw an initial embedding of a given knot or link onto a suitable handle-body in the first place. For example, there exist Borromean embeddings on a genus-3 surface, where there is only a single trace passing through each hole or winding around a handle (Figure 15c). This naturally leads to the large open question of how many fundamentally different embeddings there are of a given knot or link that cannot be turned into one another by smoothly deforming the host handle-body.

With respect to making visible the effect of everting a handle-body with an embedded knot or link, one reviewer suggested a way how this might be done virtually by switching the point of view from a place outside the original handle-body to a place inside: Start with the handle-body embedded in R^3, and choose some point P "inside" the object. Compactify R^3 with a single point at infinity to form S^3. Remove P from this S^3 to obtain another copy of R^3. In order to view the object from within that R^3, one might computationally perform an inverse stereographic projection to go from R^3 to S^3, then rotate the S^3 to position P where infinity was, and finally use a stereographic projection again to get back to R^3. This looks like a nice project and follow-up paper for someone with the necessary computer-graphics skills!

Turning Visualization Models into Sculptures

Eversions of embedded knots or links are difficult to understand. A variety of different 3D models can help in this quest. The question then arises, whether some of these explanatory models can be refined and enhanced in their aesthetic appeal, so that they stand on their own as small sculptures, even if their mathematical significance is ignored.

One promising approach uses a handle-body that recedes in importance to make the embedded ribbons stand out most prominently. In Figure 15a, the *Tetrus* handle-body has been rendered as a partly transparent, gridded surface, supporting solid ribbons representing the Borromean rings. Such a model could also be fabricated on a more expensive 3D printer, where the handle-body and the ribbons can be built from different materials with different translucencies. Alternatively, the handle-body could be fabricated on an inexpensive 3D-printer (Figure 15b), and the link-traces could be represented by ropes or by pipe-cleaners, which are physically wound around this body. Figure 15c shows a resulting sculpture model of a different embedding of the Borromean rings on a 3-handle *Stem* structure; in this embedding each handles is used by only a single ring trace. This allows me to wind that trace an arbitrary number of times around the handle without changing the topology of the Borromean link.

(a) (b) (c)

Figure 15: *Sculpture models of the Borromean rings: (a) on a Tetrus, and (b,c) on a 3-handle Stem.*

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References

[1] A. Chéritat. "Torus eversion." *Video*, June 24, 2017. https://www.youtube.com/watch?v=UARcfHZJmmI

[2] L. Hsu, R. Kusner, J. Sullivan. "Minimizing the Squared Mean Curvature Integral for Surfaces in Space Forms." *Experimental Mathematics*, Vol.1, No.3, 1992. http://torus.math.uiuc.edu/jms/Papers/willmore.pdf

[3] Mathologer. "Smale's inside out paradox." *Video*. 2016. https://www.youtube.com/watch?v=ixduANVe0gg

[4] G. McShane. "Torus eversion." *Video*. Nov. 24, 2008. https://www.youtube.com/watch?v=S4ddRPvwcZI

[5] A. Phillips. "Turning a Surface Inside Out." *Sci. Amer.* **214**, 112-120, Jan. 1966.

[6] "The Rolfsen Knot Table." http://katlas.org/wiki/The_Rolfsen_Knot_Table

[7] C. H. Séquin. "Physical Illustrations of Knot Deformations and Surface Eversions." *ICERM*, Brown University, Sept. 18, 2019. http://people.eecs.berkeley.edu/~sequin/TALKS/2019/2019_ICERM-Talk.ppt

[8] StackExchange. "Mathematics." 2018. https://math.stackexchange.com/questions/2612627/involution-of-the-3-and-4-holed-torus-and-its-effects-on-some-knots-and-links

[9] J. M. Sullivan. "A History of Sphere Eversions." 1999. http://torus.math.uiuc.edu/jms/Papers/isama/color/opt2.htm