Dürer Machines Running Back and Forth

António Bandeira Araújo

Universidade Aberta, Lisbon, Portugal; antonio.araujo@uab.pt

Abstract

In this workshop we will use Dürer machines, both physical (made of thread) and virtual (made of lines) to create anamorphic illusions of objects made up of simple boxes, these being the building blocks for more complex illusions. We will consider monocular oblique anamorphosis and then anaglyphic anamorphoses, to be seen with red-blue 3D glasses.

Introduction

In this workshop we will draw optical illusions – anamorphoses – using simple techniques of orthographic projection derived from descriptive geometry. The main part of this workshop will be adequate for anyone over 15 years of age with an interest in visual illusions, but it is especially useful for teachers of perspective and descriptive geometry.

In past editions of Bridges and elsewhere at length [1, 6] I have argued that anamorphosis is a more fundamental concept than perspective, and is in fact the basic concept from which all conical perspectives – both linear and curvilinear – are derived, yet in those past Bridges workshops I have used anamorphosis only implicitly, in the creation of fisheye [4] and equirectangular [3] perspectives . This time we will instead focus on a sequence of constructions of the more traditional anamorphoses: the so-called oblique anamorphoses, that is, plane anamorphoses where the object of interest is drawn at a grazing angle to the projection plane.

This sequence of constructions has been taught at Univ. Aberta in Portugal since 2013 in several courses to diverse publics; it is currently part of a course for Ph.D. students in digital art media, where it serves as an introduction to the mathematics behind digital tools such as VR panoramas and 360-degree video [2], but it started out in 2013 as a course for school teachers. I believe that these activities are very useful for young students just beginning their study of both linear perspective and descriptive geometry. They have been tried with several groups of young students, mostly Portuguese 9th-grade classes (around 15 year-olds) with very satisfying results. Starting the learning process of perspective with anamorphosis is both impressive to the students (who are surprisingly in awe of this primitive, analog sort of VR) and theoretically sound, paving the way to an understanding of perspective in its more general form. It has also the advantage of at the same time constituting a natural exercise in descriptive geometry. The constructions use the fundamental notions of descriptive geometry in a setting that is enticing to students and conceptually interesting, with results that are visually impressive and eminently “instagrammable”, since anamorphoses of such small scale are hard to see with the naked eye but come immediately alive when shot by the camera of a mobile phone. This creates an impressive effect joining the charming magic of physical constructions with the now more familiar world of smartphone visuals.

One of the difficulties of teaching descriptive geometry is that it is hard for a student to verify the correctness of his results unless he already knows descriptive geometry – therefore the student needs an “oracle” (the teacher) to verify correctness of an exercise. Anamorphoses work as descriptive geometry exercises that dispense with this oracle, since the efficacy of the end result as a visual illusion is evidence of the mathematical correctness of the construction. Students appreciate this clear visual confirmation – and reward – to the correctness of their work.

Principles

Conical anamorphosis derives from a single notion, which I call the principle of radial occlusion [1]: two points on the same ray from O will look the same to an observer at O.

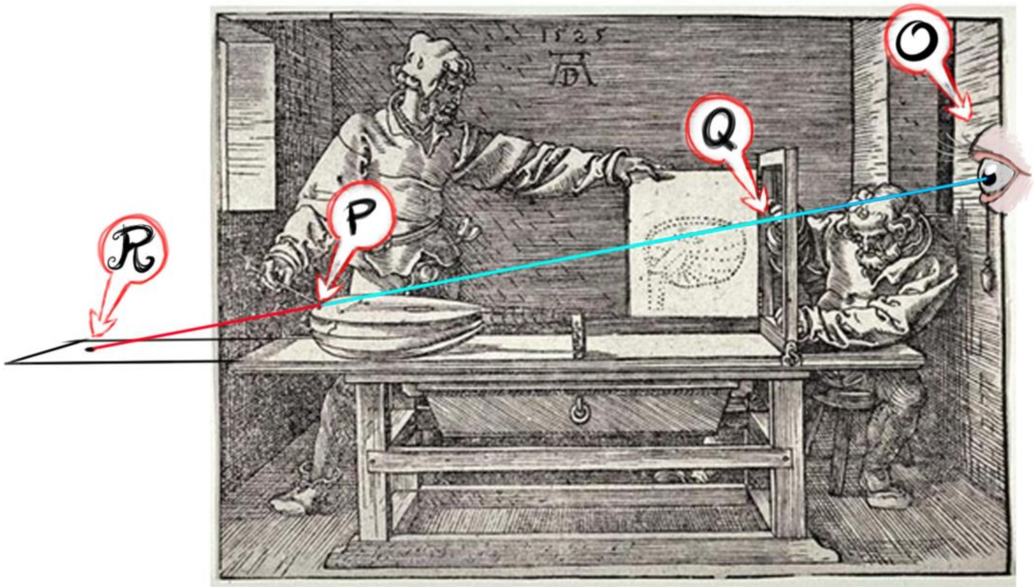

Figure 1: *Dürer's perspective machine can be run as a push-forward rather than a pull-back device, thus becoming a machine for oblique anamorphoses. Point P on the object is visually equivalent to both point Q (perspective) and point R (oblique anamorphosis), all three points being anamorphs of each other since they lie on the same ray from O. Original print by Albrecht Dürer, defaced by the author.*

We say that two objects are *anamorphs* relative to O if they subtend the same visual cone from O. The principle of radial occlusion implies that two 3D objects will look the same from O if they are anamorphs. In particular a 3D object will be visually indistinguishable from the intersection of its visual cone with a 2D surface: that is why the illusion of realistic drawing is possible. If you think about it, it is quite surprising that a 2D object should pass for a 3D one, but this is a simple consequence of accepting radial occlusion as valid.

I would argue this principle of equivalence has a very old ancestry, being the proper theme of Euclid's Optics [8]. This was long misunderstood, by Panofsky [14], Goodman [11] and Kline [13], among others, who, as Brownson [7] discusses, thought Euclid's Optics incompatible with linear perspective. None of this of course interfered too much with the practical uses of anamorphosis over the years by its masters such as Pozzo [9] or Niceron [15] or Robert Barker [12] for that matter, who used it to its full extent as a precursor to what we would now call the virtual or mixed realities [12, 16].

From radial occlusion follows a multitude of consequences: that the natural manifold of visual data is the sphere, not the projective plane; that the natural definition of vanishing points should attribute exactly two of them to each line. These consequences have been explored elsewhere [5]. Here we will consider only a simple practical construction of anamorphs: the Dürer machine. In Figure 1 we see a Dürer perspective machine that we have subverted into constructing so-called oblique anamorphosis. The usual purpose of this machine is to build a perspective drawing on the vertical plane, by identifying point P in the object to be drawn with a point Q in the picture (see Figure 1). But by the principle of radial occlusion, point R on the table is just as valid an anamorph of P. Hence, running Dürer's machine forth onto the table rather than back to the vertical canvas should produce a drawing just as convincing as a visual illusion, even if imbued with a much different metric deformation.

In this workshop we will exploit this generalization of Dürer's machine, using the real machine, with thread and fixed point, as a concrete guide, and then abstracting it into descriptive geometry constructions.

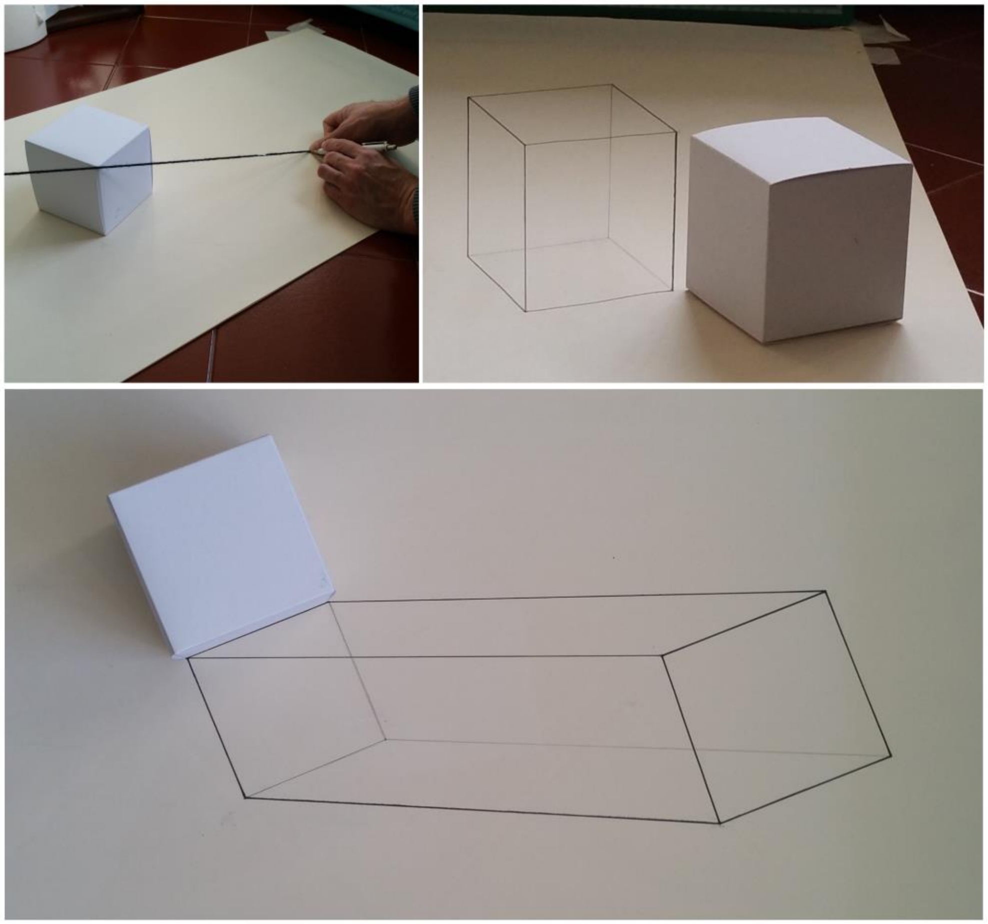

Figure 2: *Anamorphosis of a cube built with a Dürer machine. The vertices of the cube, projected from O through a thread (upper left) define a "deformed" drawing (seen from above in the lower picture) that looks like the original cube when seen from O (upper right).*

Activities

The workshop will be split into a sequence of practical and theoretical activities. The number of participants should not exceed twenty. All the necessary materials will be provided by the instructor.

The activities should be enjoyable to people with very moderate drawing skills and with even a casual recreational interest in optical illusions, but it should be especially interesting to teachers wishing to instruct young students just starting out their study of perspective, or to motivate their study of descriptive geometry. Because it is important to transmit a more general framework to such teachers, while being restricted to the 1.5 hour format of the workshop, we will perform certain tasks only partially, explain others theoretically, and choose only a few to execute thoroughly, as drawing anamorphoses can be a very time consuming affair. The most drawn out construction will be that of the orthographic piece (item 2 of the activities discussed ahead), the rest of the workshop piggybacking on that construction through alterations and additions that exemplify the further construction under discussion.

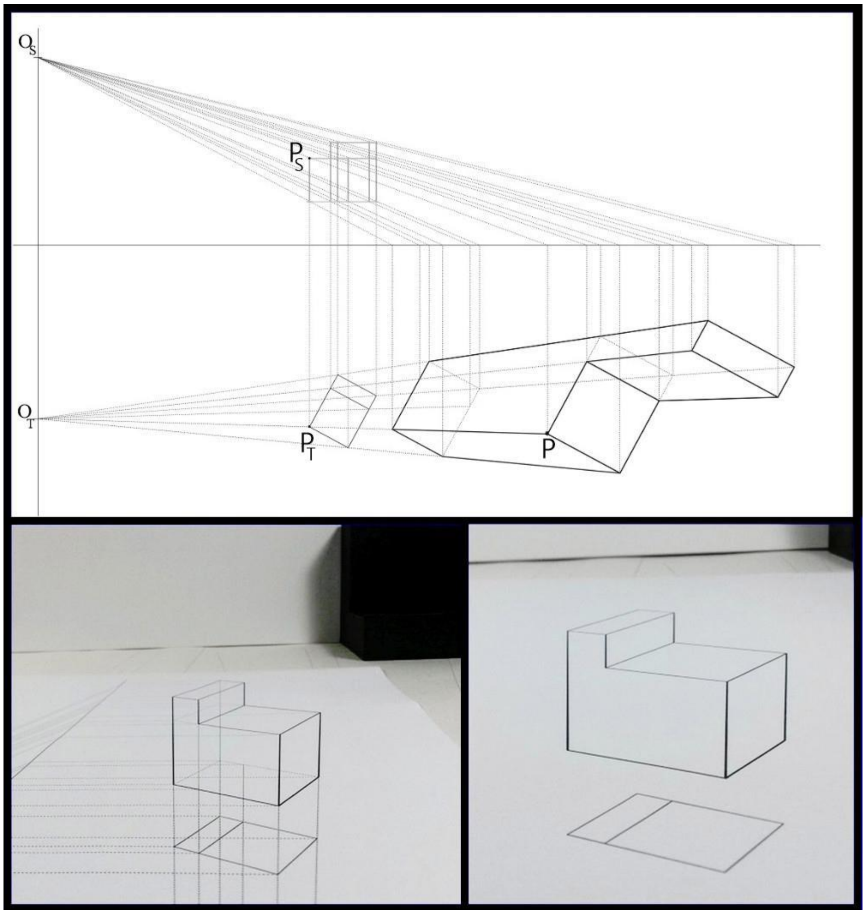

Figure 3: *Orthographic implementation of a Dürer machine. Student exercise by Manuel Flores.*

Our activities will be as follows:

1) We will first implement a physical Dürer machine, using a thread fixed at a point O on top of a tripod: the thread passes through each vertex of the cube in turn; where it hits the floor, points are marked (Figure 2, top left), then joined to obtain a metrically deformed projection (Figure 2, bottom) that will however look like the original cube when seen from O (Figure 2, top right). The first steps will be done by the instructor, the next by the participants.

2) I will then show how the Dürer machine may be abstracted as an orthographic construction in descriptive geometry, by drawing a side and top view of the physical setup, and tracing on this the threads trajectory in projection. Then each group of students will make their own orthographic drawing of a simple anamorphosis of a basic object, like a cube, or an L-figure, or some other simple object made up of one or two boxes. In Figure 3, we see a orthographic construction of an L-shaped object. First, the top view of the object has been drawn freely, at some desired position relative to the top O_T view of the observation point O. Then this drawing is raised to the side view above it, where the side view O_S of the observation point is placed at a desired height above O_T, thus determining O in 3D space. To find the anamorphic image of a vertex P of the object with side view P_S and top view P_T we do as follows: 1) mark where O_SP_S hits the ground line; at this point trace a vertical through the top view. Where that vertical intersects O_TP_T the anamorphic projection of P is found on the top view. Join the vertices thus obtained. The resulting image found on the top view is the anamorphosis of the cube, to be later viewed or photographed from above point O_T by a height determined by O_S.

This is the main practical task, that each group of participants will complete in its entirety. We will then verify the appearance of the anamorphosis by photographing it with mobile phones. The long depth of field of these small phone cameras is excellent to render anamorphoses, as they are quite similar to the ideal monocular eye of perspective.

3) We will now reason over the constructions we made. We will see how the projections determine the location of the observation point in 3D space. We will discuss the location of vanishing points in the anamorphic construction. We will see that the vanishing points of a line *l* are the intersection with the projection surface of the line parallel to *l* through *O*. We will identify these points by physically pointing in the direction of *l* while standing over our drawing. Regarding our example, we will see that the vanishing point of verticals will be the point on the floor under the observer, and that the images of horizontals will be parallel to each other on the floor; for instance, the lines that look like parallel verticals in Figure 3 (bottom) are seen to converge on Figure 3 (top) at O_T, the point on the projection plane that lies under the observer's eye. We will also see how moving the observer on a vertical over the vanishing point of the boxes verticals will cause controlled deformations of the anamorph, resulting for instance in a cube stretching vertically in appearance while remaining otherwise undeformed in appearance (a striking effect quite similar to an animated bar chart – see Figure 4 and the video example at https://youtu.be/YogNi52ycEU). Finally, we will discuss in what scenarios an anamorphic drawing might have two vanishing points for some lines (for instance, if the projection surface is the whole room) and what the line projections would look like as they go to their twin vanishing points.

4) We will discuss perspective arithmetic: how to subdivide and multiply the box we have drawn with internal constructions of diagonals and lines sent to vanishing points. This will be exemplified by the instructor in one of the drawings made by the participants. Subdivision and multiplication of anamorphic cubes allows for the construction of arbitrarily complex forms, through gridding and interpolation.

5) We will discuss how to color an anamorphosis in a credible manner, not only intuitively (Figures 4 and 5), but relating it to theories of light and color. Namely we will discuss the Lambert model of illumination [10], and how the gray value relates to the cosine of angle of incidence of the light (for matte objects).

6) We will consider the construction of binocular anamorphoses, by making an anaglyphic cube such as that of Figure 6 (left). This requires a doubling of our cube drawing, repeating the process for an observation point O' at the interocular distance from the initial observation point *O*. Each drawing must be colored red or blue, and the result composite, a 3D anaglyph, will be observed through red and blue 3D glasses, thereby serving each eye with only the drawing done from its corresponding position. The resulting anaglyph will seem to jump from the page, creating a visual effect akin to an augmented reality construct. Waving the head slightly sideways creates a striking effect, as if watching a computer wireframe drawing floating midair over the physical scenario of the room.

7) Finally we will briefly discuss more complex constructions of anamorphoses that serve as pretext for more intricate descriptive geometry exercises. For instance in Figure 7 we see the descriptive geometry construction necessary for obtaining the anamorphic image of a single point M (say, a vertex of our cube) onto an oblique plane. This is an excellent pretext to learn the intersection of a line with a plane and of a ramp with an oblique plane, two standard exercises of descriptive geometry. It could be used to make an anamorphic illusion on a board cut to fit obliquely into a corner of a room.

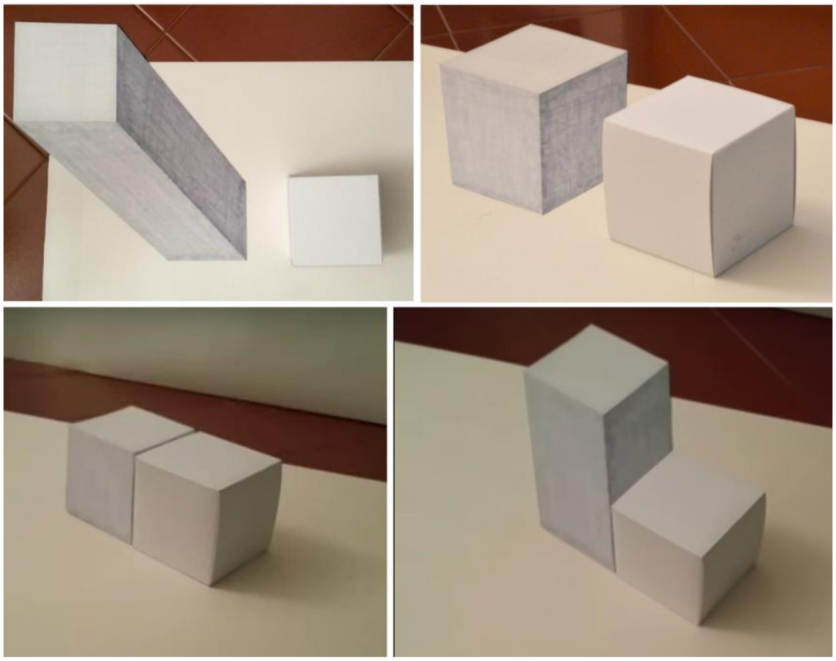

Figure 4: *Top left: a solid cube and its flat anamorph. Top right: The view from O, where the anamorph looks the same as the real cube. The flat anamorph is painted (in grey markers) in such a way as to simulate the values of the real cube under the prescribed light. Bottom: If the point of observation O is raised along a vertical line, the vanishing point of the cube's verticals will not change, resulting in a motion akin to that of a rising bar in a 3D bar chart. Drawing by the author.*

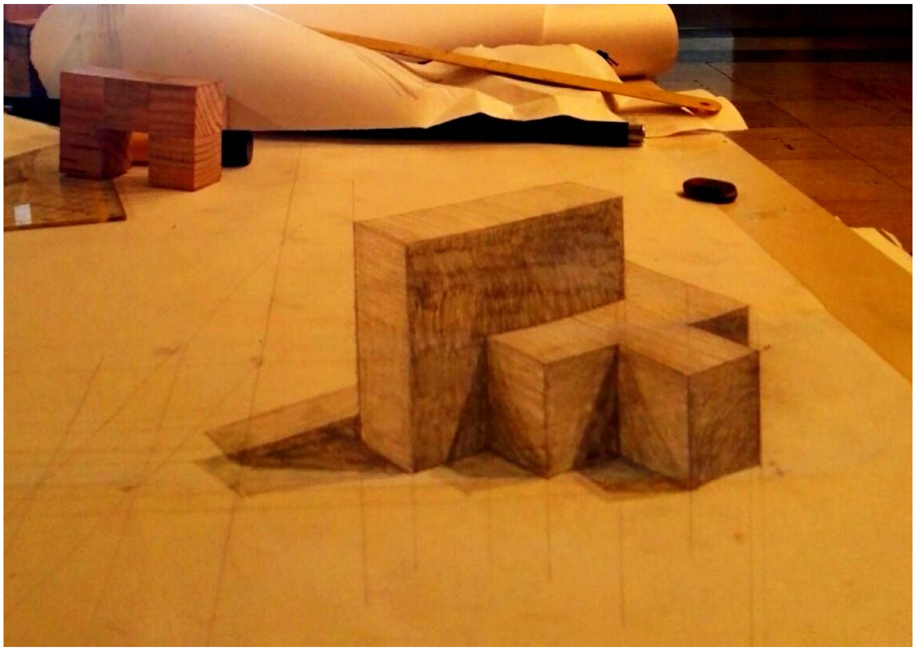

Figure 5: *Another example of an exercise made with the orthographic method, this time involving not just value but also multiple projected shadows. Exercise drawing by Maria Bianchi Aguiar.*

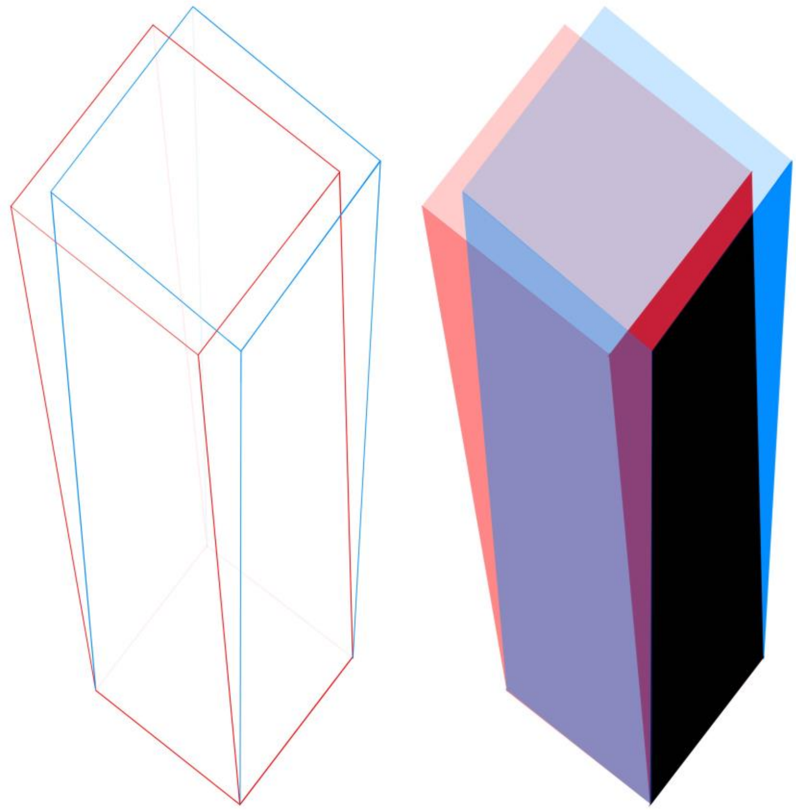

Figure 6: *Binocular/Anaglyphic anamorphosis of a cube. Left: wireframe version. Right: solid version. Seen with with red-blue 3D glasses from the correct point, at a grazing angle to the projection plane, these will seem to pop out of the plane of the paper. You can try filling a screen with one of these (a 12" screen will suffice) or an A4 printed sheet, and you should be able to see the illusion. Note that Red Cyan glasses will not work as well as the older Red Blue types as they create distracting ghosting effects.*

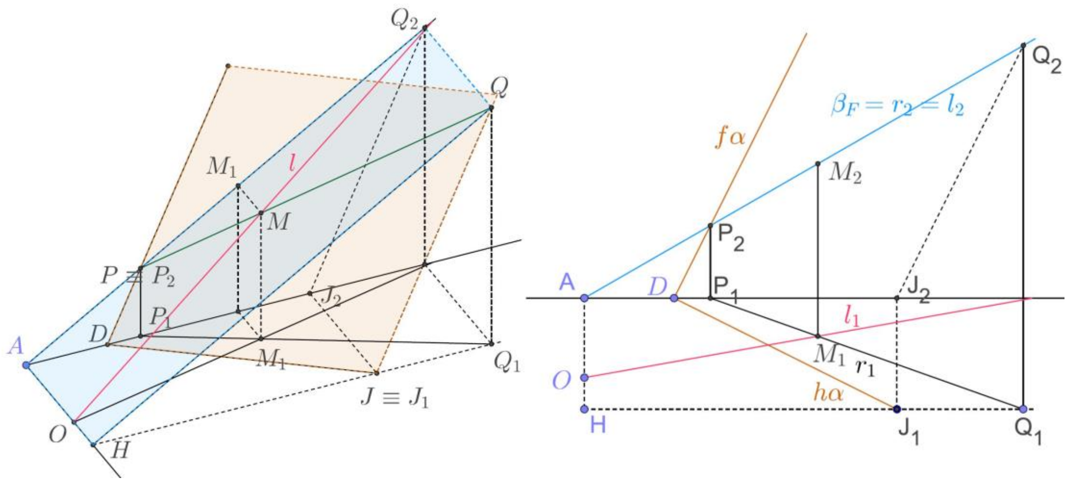

Figure 7: *Using descriptive geometry to plot the anamorphic image of a point M relative to the observer at O, on the oblique plane DJQ (brown). Left: 3D view. Right: Side and Top views. First a ramp plane (blue) is intersected with DJQ to find line PQ(green), then M is found by intersecting PQ with the vertical plane through l, which projects as line l_1(red) in the top view.*

Summary and Conclusions

We investigate construction methods for conical anamorphoses of simple box-like forms, using basic descriptive geometry. These forms in turn can be multiplied or subdivided to generate arbitrarily complex anamorphic constructions, through interpolation and gridding. The methods presented can be used for recreational or artistic purposes, or as a motivation for the study of descriptive geometry and a foundation for the study of perspective.

Acknowledgements

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