Near-miss Star Patterns

Paul Gailiunas

paulgailiunas@yahoo.co.uk

Abstract

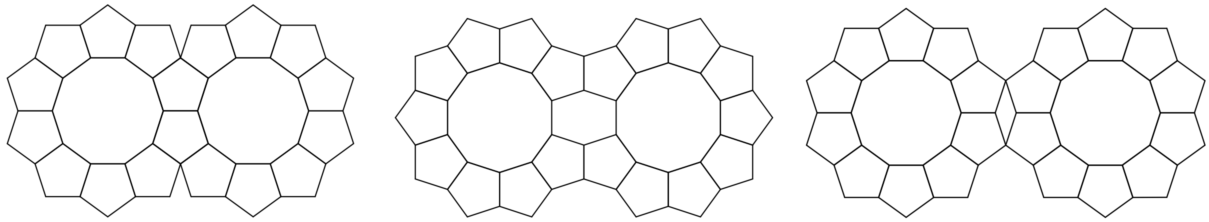
Most Islamic geometric star patterns can be derived from tilings of regular polygons, and often additional non-regular polygons are needed, especially in designs based on pentagons/decagons. Generally the angular relationships between the polygons are exact, but there are a few examples, mainly Mamluk with a few from the Seljuk Sultanate of Rum, that depend on approximations. Variations in the additional interstitial polygons accommodate the inaccuracies. Many more such possibilities exist, and a fairly simple spreadsheet technique allows for their investigation.

The Polygonal Technique

A few different methods have been used to construct the star patterns generally associated with Islamic culture, but the most versatile begins with a polygonal tiling. At its simplest the tiling is uniform or possibly k-uniform [4, pp. 58-69], but it is also common, sometimes necessary, for non-regular polygons to be included [3, pp. 221−253]. The tiling, or more precisely the mid-points of its edges, provides a template for the creation of star patterns that are constructed by drawing lines at each mid-edge that intersect at an angle that is characteristic of the pattern (Figure 4(d)). The constructed lines determine paths that turn where the lines meet, so that generally paths have alternating crossing points (at the edges of the tiling) and turning points, although there can be exceptional situations when a path goes through crossing points without an intervening turn. There are historic examples with paths that are thick coloured lines, possibly outlined so that they seem to weave over and under, or the paths determine the edges of coloured tesserae. The crossing angle relates to the underlying tiling, so that, for example, a pattern derived from (3.6.3.6) [4, p. 63] might have crossings at 30°, and paths on a one with decagons could cross at 108°.

While simple examples are essentially decorations of an underlying uniform tiling, it is more natural to consider more elaborate templates as *primary polygons*, which determine the visually dominant stars in the final pattern, embedded in a field that is filled with smaller *interstitial polygons*. Patterns based on decagons (10-gons) as primary polygons provide a paradigm, since two regular pentagons and a regular 10-gon will fit around a single point, although the arrangement cannot be extended without leaving gaps, so other non-regular polygons are necessary. Two 10-gons can be arranged vertex to vertex as in Figure 1(a) or edge to edge as in Figure 1(b), but a vertex to edge arrangement does not provide a satisfactory template. Wider separation is possible, for example Figure 1(c), although some modification might be needed to accommodate the rhombus.

Primary polygons other than 10-gons can be used in a similar way but the pentagons will not be regular. The lines of symmetry of the primary polygons, which collectively constitute what Bonner calls a “radii matrix” [3, p. 378], are the basis of such designs. In particular neighbouring primary polygons share a line of symmetry, either through vertices (Figure 1(a) and (c)) or an edge (Figure 1(b)), and lines of symmetry determine three edges of common pentagons (a), or four edges of a common hexagon (b).



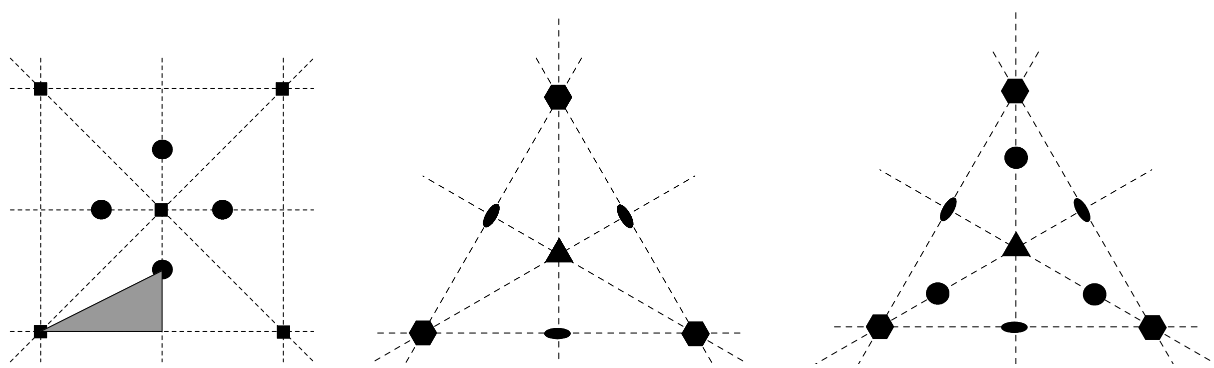
(a) (b) (c)

Figure 1: *Decagons can be surrounded by regular pentagons but other irregular interstitial polygons are needed to extend the tiling.*

Locating the Primary Polygons

Assuming that the pattern is to be periodic it will have the symmetry of one of the 17 wallpaper groups. The centres of primary polygons are located most naturally on points of rotational symmetry, and the vast majority of historical examples use no other positions. The polygon must share the symmetry of the global design so that, for example, if an n-gon is placed on a centre of 6-fold rotation then n must be divisible by 6. In principle there might be another design that has primary polygons in the same locations but with n odd, but of course the pattern would have lower three-fold symmetry.

Since all regular polygons have lines of mirror symmetry there is no such restriction in principle on what can be located on global lines of symmetry. For example 11-gons [1] or 13-gons [2] are known historically. Some symmetrical examples have this polygon on a mirror-line that is perpendicular to another one (Figure 2(a)), so there is a right-angled triangle (shaded) with vertices at the centre of the polygon and a centre of rotation e.g. [3, pp. 456 - 457, fig. 399]. Mirror-lines that intersect at right-angles induce a 2-fold rotation, so any n-gon located there must have n even (Figure 2(b)). Many such examples are known [e.g. 3, pp. 425 – 428, figs. 362 - 366], and there is one with the arrangement shown in Figure 2(c) that has polygons on the line joining centres of rotation [3, pp. 429 – 430, figs. 369 - 370].



(a) (b) (c)

Figure 2: *Some possible locations for primary polygons. Polygons indicated by a circle need only be regular; those indicated by an ellipse must have an even number of sides. Polygons at centres of rotation must share that symmetry: 4-fold in (a); 3-fold and 6-fold in (b) and (c).*

Finding Likely Polygons

Since designs require neighbouring primary polygons to share a line of mirror symmetry, the geometry of arrangements such as those in Figure 2 restricts the polygons that can be used, but there are historical examples that cannot obey the restrictions. For example the Topkapı scroll includes the arrangement in

Figure 2(a) with 16-gons at the corners and 13-gons on the mirror-lines [2], so the corresponding right-angled triangle cannot be exact. In fact the hypotenuse and one side pass through adjacent vertices of the 16-gon, making an angle of 22.5°. It will be more convenient to work in fractions of 360°, so this angle is 1/16 (one "step" around the polygon). The angle at the 13-gon is 5/26, between a vertex and the mid-point of an edge (2½ steps around the polygon). The two angles add to 53/208 rather than 54/208 as required, and actually the "hypotenuse" is not a straight line, but the error is so small that it is not visible. Several such "near-miss" designs are known, but more can be found.

A spreadsheet allows quite a simple approach. The Farey sequence of order N lists the fractions in lowest terms (any common factors are cancelled) between 0 and 1, with denominator not greater than N, in order of size. There is a standard recurrence relation to generate the terms of this sequence: if successive terms in a sequence of order N are a/b and c/d then the next term has a numerator = $c \times \text{int}((N + b)/d) - a$ and denominator = $d \times \text{int}((N + b)/d) - b$ (where int means integer part). Putting the first term as 0/1 and the second as 1/N the recurrence relation can be used to calculate subsequent terms along two rows of a spreadsheet. If the rows are copied into two columns then the fractions can be used to label an array of cells that calculate their sum. Conditional formatting can then be used to highlight the cells that hold a sum exactly equal to a target or within a specified range from it (supplementary material).

Since the polygons can be either edge/edge or vertex/vertex it is convenient to use a target that is double the required value: ½ rather than ¼ for a right-angled triangle. The cell (indicated in Figure 3) corresponding to the design quoted above is labelled 1/8 (= 2/16), 5/13. It follows that the design has 16-gons (8-gons might be possible) and 13-gons. The required angles need half of the numerators: 1 step around the 16-gon (or ½ step if an octagon were used) and 2½ steps around the 13-gon.

1	target		1																			
2			2	0	1	1	1	1	1	1	1	1	2	1	2	1	2	1	3	2	3	1
3	±		0.012	1	15	14	13	12	11	10	9	8	15	7	13	6	11	5	14	9	13	4
4	1	0	0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	1	4	0.3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
24	4	15	0.3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
25	3	11	0.3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
26	2	7	0.3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
27	3	10	0.3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
28	4	13	0.3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
29	1	3	0.3	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
30	5	14	0.4	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
31	4	11	0.4	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
32	3	8	0.4	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1
33	5	13	0.4	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1
34	2	5	0.4	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
35	5	12	0.4	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
36	3	7	0.4	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

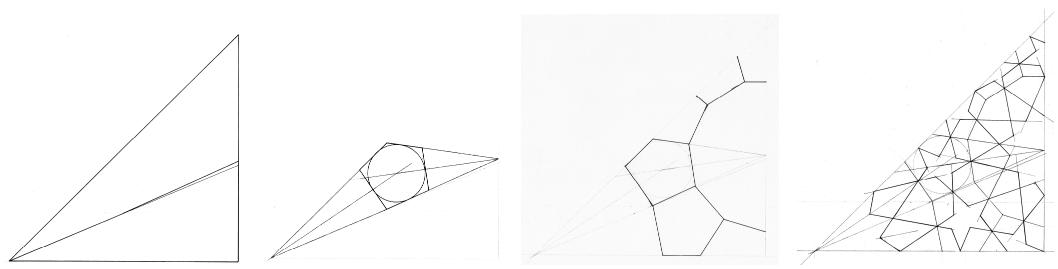
Figure 3: *A spreadsheet showing exact solutions (dark) and near misses (light) for the angles in a right-angled triangle. The actual values (here rounded to the nearest integer) are not needed.*

An Example

Consider a similar case with one of the primary polygons located on a centre of 4-fold symmetry: it must also have 4-fold symmetry, so not all approximations can be used. There is a further consideration: the other primary polygon that lies on a mirror-line should not be too near to other centres or lines of symmetry, which might cause overlap. In particular it should not be too close to its mirror image. The angle we need should not be too far from 1/16 (see Figure 2(a)), and the other possibilities, 1/12 and 1/20, are unlikely to be satisfactory without a union or intersection of overlapping polygons. It happens that an 11-gon provides another approximation that is almost as close as the 13-gon (in the same column in Figure 3). There is also an exact solution corresponding with the (4.8^2) tiling.

The symmetry of the finished design will be ***442** (or p4m in IUC notation), and the fundamental domain is an isosceles right-angled triangle, so the initial constructions can be restricted to this region. The first stage is to locate the primary polygons. The 16-gon will be centred at one of the 45° vertices. An approximate position of the 11-gon can be found by drawing the line that will pass through a vertex of the 16-gon, i.e. at 22.5° to the side of the triangle, as in Figure 4(a). The centre of the 11-gon could be located where this line cuts the other side of the triangle. In many cases this will be good enough, but a better location is found by marking the mid-point of the line and drawing another line from it making an angle of 2/11 (= 65.45°) with the vertical side of the triangle. The difference is hardly visible.

The next stage is to determine the size of the primary polygons. Bonner has described a method when the arrangement is like Figure 1(a) [3, pp. 378-379]. Construct the incircle of the triangle formed by the (slightly crooked) line joining the centres already located and the lines that must pass through the next vertices (making angles of 1/16 and 1/11). Its centre lies at the intersection of the lines through the mid-points of the edges. Tangents at the points where these lines cut the incircle are the required polygonal edges (Figure 4(b)). The other edges and the rest of the interstitial polygons (all pentagons in this case) can be completed since their edges are defined by lines of symmetry of the polygons (Figure 4(c)).



(a) (b) (c) (d)

Figure 4: *Stages in constructing a star pattern.*

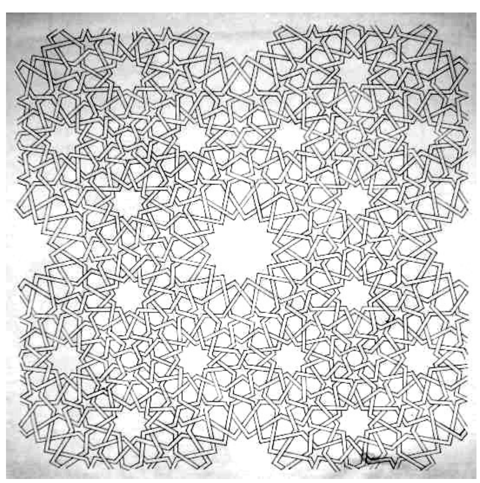


Figure 5: *The finished design with 16-gons and 11-gons.*

Pairs of lines can now be constructed, crossing at the mid-points of the edges of all the polygons. In this case a crossing angle of 45° has been used, since it is consistent with the overall symmetry as well as with one of the primary polygons (Figure 4(d)). An angle associated with the 11-gon is unlikely to be as aesthetically successful. Figure 5 shows the design that has been completed with strapwork following the paths in Figure 4(d) to create an interweaving effect. The supplementary material includes images of two other new designs. One, with 18- 15- and 11-gons, uses the same idea as Figure 2(a) applied to a tiling of hexagons. The other, based on a rectangle having 10-gons at its centre and corners, is more complicated and needed a lot of adjustment and compromise, but it gives an idea of what is possible.

The Problem of Size

The example described has only two types of polygon and two relationships to consider: 16-gon/11-gon, which is the basis of the construction, and 11-gon/11-gon, which was controlled by looking for locations about half-way between the mirror-line and centre of rotation. Things are not so simple if there are more primary polygons such as the arrangement in Figure 2(c), with more relationships. There are historical examples of this arrangement, which Bonner has argued are the most complex in the tradition [3, pp.428-430]: in the courtyard portal of Seri Han near Avanos (1230-1235) and at the entry to the mosque at Karatay Han near Kayseri (1235-1241). It is the same design in both cases with 9-, 10-, 11- and 12-gons. For the purposes of discussion it will be convenient to number the four types of polygon, and the symmetry requirements provide a mnemonic:

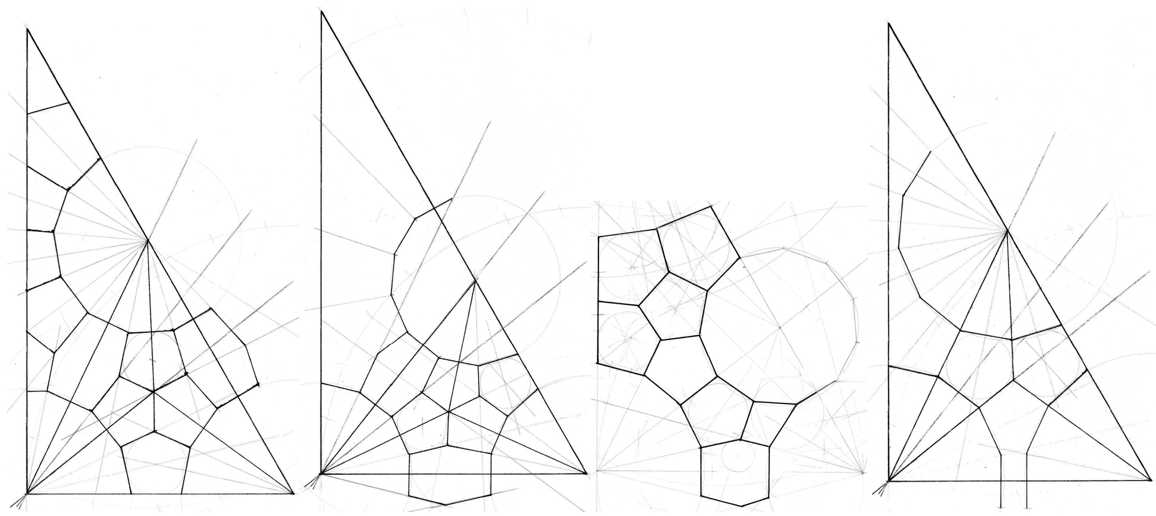
1. Polygons marked with circles in Figure 2(c), on the mirror-line joining 3-fold and 6-fold centres of rotation, with no symmetry requirement (11-gons in the historic examples).
2. Polygons marked with ellipses, at the intersection of two mirror-lines, which must have 2-fold symmetry, so with an even number of sides (10-gons in the historic examples).
3. Polygons marked with triangles, at centres of 3-fold symmetry, with the number of sides divisible by 3 (9-gons in the historic examples).
4. Polygons marked with hexagons, at centres of 6-fold symmetry, with the number of sides divisible by 6 (12-gons in the historic examples).

Suitable polygons can be found by considering the triangle defined by the centres of polygons 1, 2 and 4. The angles at centres 1 and 2 must sum to 5/12 (or 5/6 if the doubling approach is used), and polygon 1 should be approximately equidistant from centres 3 and 4. There is an exact solution that locates polygons 1 and 2 on the edges of (3.12^2) that might be worth further investigation. As well as the historical design there are three near-misses with polygon 2 being a 14-gon, two with 13-gons and another with 11-gons, with the approximation particularly close in two cases.

Each pair taken from polygons 1, 2 and 3 must connect either with pentagons or hexagons, as in Figure 1. There are common lines of symmetry between the polygons (approximate between 1 and 2) but there should also be three lines, one mirror-line from each polygon, that are approximately coincident (see Figure 6). Consider the point of intersection of mirror-lines (compare with Figure 4(b)) from polygons 1 and 2. A line from this point to centre 3 should be a mirror-line of polygon 3, and it will determine the number of its sides. In the historical example the line is at an angle very close to 20° to the line from 2, so polygon 3 is a 9-gon. It turns out that all three of the 14-gon cases have suitable angles: (using steps) in 4½/13, 1/14 it is close to 36° (= 1½/15); in 4/13, 1½ /14 it is close to 24° (= 1/15); and in 3/11, 2/14 it is close to 15° (= ½ /12), giving two possibilities with 13-, 14-, and 15-gons, and another with 11-, 14-, and 12-gons. The vertex/vertex or edge/edge requirement of Figure 1 is satisfied in all three arrangements, so six arrangements are possible in principle.

In all of the 14-gon possibilities the angle relationships work out surprisingly well, but problems arise with the next stage: determining the size of the primary polygons using the incircle construction of Figure

4(b). Figure 6(a) shows the only straightforward case: 4½/13, 1/14, 1½/15, 1/12 with the 14- and 15-gons vertex to vertex. Figure 6(b), 4/13, 1½/14, 1/15 with the 13- and 15-gons vertex to vertex might be made to work with some additional interstitial polygons. In Figure 6(c), 3/11, 2/14, ½/12, 2/24 with the 11- and 14-gons vertex to vertex, the 11-gon and 12-gon are (almost) touching and there is an interstitial quadrilateral. In figure 6(d), which has the same polygons as 6(a), but with the size fixed by the vertex/vertex relationship between the 13-gon and 14-gon, the space between the 14- and 15-gon is too small for a workable hexagon, and the 13-gon is too close to the mirror-plane, and hence another 13-gon.



(a) (b) (c) (d)

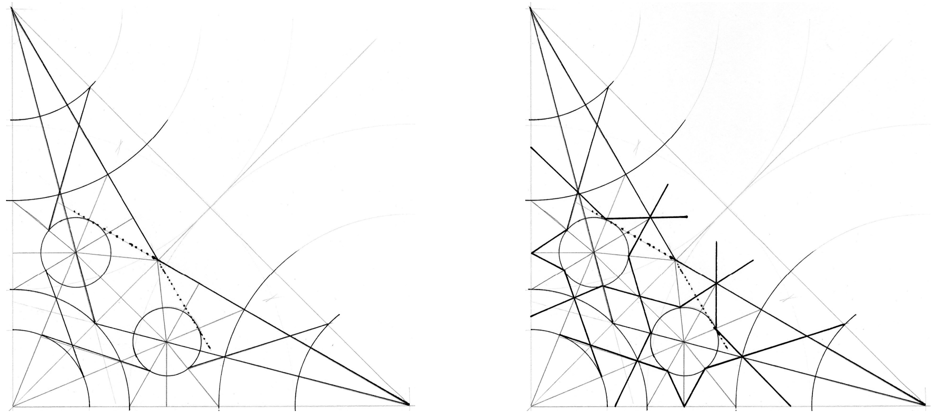
Figure 6: *Some polygonal tilings created from arrangement 2(c) with 14-gons.*

Interstitial Polygons

While in these near-miss designs the primary polygons are always regular, the interstitial polygons never are. Pentagons occur most frequently, and they can be regular only if they lie between two decagons. The incircle construction provides a method that produces what might be a good approximation to regularity, but it cannot be used consistently in even the simplest cases. In Figure 5, for example, the sizes of the underlying primary polygons have been chosen so that the pentagons between 11- and 16-gons have incircles. Since the 11-gons are now determined, so are the pentagons that lie between them, and they do not have incircles. More complicated cases have correspondingly more incompatibilities, which can lead to unworkable situations, such as that in Figure 6(d).

In fact the incircle construction does not appear in historical documents such as the Topkapı scroll. Circles are used, but, rather than determining polygons from which the intersecting lines are subsequently constructed, they directly define the intersecting lines that form the stars. Catalog Number 35 [7, p. 252] is particularly revealing (Figure 7). The basis of the design consists of four 12-stars placed symmetrically around an 8-star. The centres of interstitial 5-stars are located at the intersections of mirror-lines of the primary stars, the size of the circumcircle of the 8-star is determined by the intersection of mirror-lines of two 12-stars, and the size of the circumcircles of a 12-star is determined by the intersection of one of its mirror-lines with one of the 8-stars (Figure 7(a)). Lines are drawn from the centres of the 5-stars to intersections nearby, and so they are not symmetrically placed at 36° to each other. There is no construction that fixes the sizes of the incircles of the stars, but the size of any one determines the size of the others. The intersections of an incircle with the rays from the centre mark turning points of the path lines, which are constructed to go through points where mirror-lines intersect with the circumcircles of the primary stars (Figure 7(b)).

This point to point construction does not depend on a tiling of polygons, and the angles of intersection of path lines are not identical, as they would be in a polygon construction. It might be significant that Hankin's constructions [5] also use incircles of stars, and he does not use the incircle construction of Figure 4(b). Although he does not state it explicitly, it seems likely that such circles were included in the construction lines he observed in Fathpur-Sikri in India that led him to his presentation in 1905.

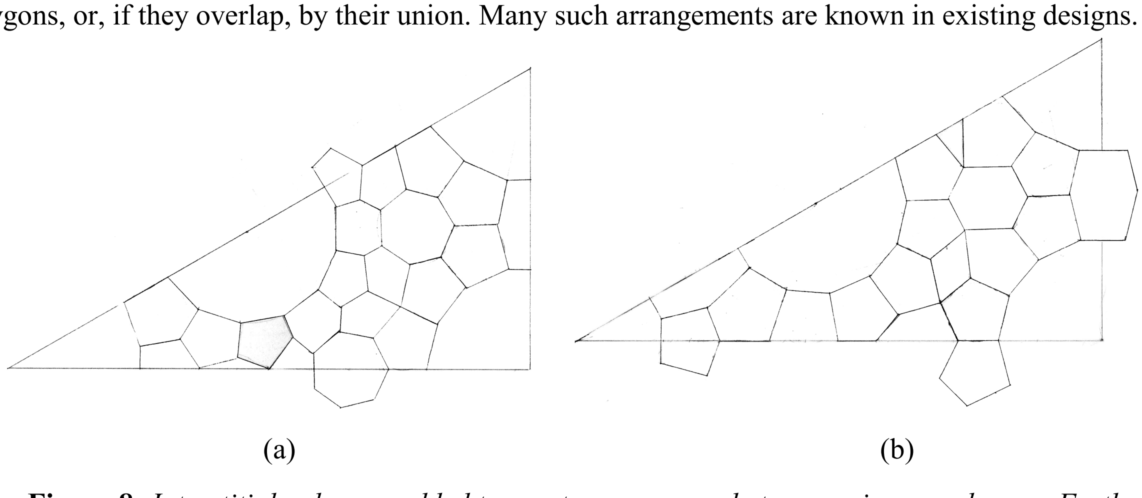


(a) (b)

Figure 7: *A design from the Topkapı scroll. (a) shows the point to point construction. The dotted extensions to the mirror lines are not present in the original. (b) shows completed 5-stars.*

Pentagons and hexagons as in Figure 1 are not the only possible interstitial polygons. Depending on the details of the pattern it can happen that gaps remain after their construction. Sometimes they work very well as additional interstitial polygons, and sometimes they must be modified [3, p. 306]. Difficult arrangements such as Figure 6(d) might be resolved by introducing addition polygons to make the primary polygons smaller (Figure 8), but this increases the number of interstitial polygons, possibly resulting in a design that is too complicated to be visually satisfying.

In arrangements like Figure 2(c), where not all main centres of symmetry are used in the near-miss arrangement, there is some additional freedom. If polygon 2 were moved along either mirror line, it would be reflected in the other. Depending on the distance moved it could be replaced by two separate polygons, or, if they overlap, by their union. Many such arrangements are known in existing designs.



(a) (b)

Figure 8: *Interstitial polygons added to create more space between primary polygons. Further modification is needed, especially in (b).*

Can Computers Help?

There are several stages in the creation of a new design using the polygonal method: locating the primary polygons in a layout, such as those in Figure 2; finding polygons with suitable angle relationships; determining the sizes of the primary polygons; completing the tiling of polygons; constructing paths that intersect on the edges of the polygonal tiling; adding detail for decorative effect. The choice of layout is probably the least susceptible to computer assistance. Polygons can be located according to taste, based on one of the wallpaper groups, or, what comes to the same thing, located at vertices of a regular or semi-regular tiling, possibly modified. The identification of suitable polygons using a spreadsheet is the central idea of this paper, and there is probably little to be gained by further automation.

The incircle construction of Figure 4(b) to determine the sizes of primary polygons works well for simpler cases, but it has intrinsic inconsistencies. Computer optimisation techniques exist precisely to manage such problems that have incompatible requirements, and could be used here, although the goal is not obvious. Should primary polygons be spaced equally, possibly weighted depending on whether they are vertex to vertex or edge to edge? Should there be a cost function related to the extent to which pentagons fail to have an incircle? Actually the polygons are not an end in themselves. Should the goal be to have stars with incircles, which seems to have directed traditional designers? Can primary polygons deviate from regularity, which is not uncommon in historic examples [1, 2, 5]? Can path lines intersect at different angles, as in Figure 7(b)?

Summary

This paper is concerned with designing new star patterns. While it is informed by evidence from historical practice, it is neither concerned with reconstructing patterns that are already known, nor with attempts to determine what methods were used in the past. Hankin's polygonal method [6] has been used because it is the only known way to draw the range of patterns made possible by this approach, but it has limitations. More powerful techniques that rely on computer-based methods are suggested, in the hope that people will be inspired to develop this form of art beyond the existing repertoire.

References

[1] B. L. Bodner. "The Unique Eleven-Pointed Star Polygon Design of the Topkapı Scroll." *Bridges Conference Proceedings*, Banff, Canada, 2009, pp. 147-154.
http://archive.bridgesmathart.org/2009/bridges2009-147.html

[2] B. L. Bodner. "The Topkapı Scroll's Thirteen-Pointed Star Polygon Design." *Bridges Conference Proceedings*, Towson, US, 2012, pp. 157-164.
http://archive.bridgesmathart.org/2012/bridges2012-157.html

[3] J. Bonner. *Islamic Geometric Patterns*. Springer, 2017.

[4] B. Grünbaum and G. C. Shephard. *Tilings and Patterns*. W. H. Freeman, 1987.

[5] E. H. Hankin. "On Some Discoveries of the Methods of Design Employed in Mohammedan Art." *Journal of the Society of Arts*, vol. 53, 1905, pp. 461-477.

[6] E. H. Hankin. "The Drawing of Geometric Patterns in Saracenic Art." *Memoirs of the Archaeological Survey of India*, No. 15, 1925.
https://archive.org/details/HankinSaracenicArt

[7] G. Necipoğlu. *The Topkapı Scroll – Geometry and Ornament in Islamic.* The Getty Center for the History of Art and Humanities, 1995.