# The Obtetrahedrille as a Modular Building Block for 3D Mathematical Art

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## Abstract

We explain how the obtetrahedrille can serve as a versatile building block for 3D structures. In particular, the obtetrahedrille can be used to construct triangular beams in the body-centered cubic lattice, and square beams in the simple cubic lattice. We present a language to describe structures composed from obtetrahedrilles. We show some sculptures that can be constructed from obtetrahedrilles. Finally, we discuss how to magnetize the obtetrahedrilles to allow simple experimentation.

**Figure 1:** *Swinger by Koos Verhoeff, polished stainless steel, Valkenswaard, Netherlands, 72 cm high (left, middle); obtetrahedrille, brushed stainless steel (right)*

*Swinger* (Fig. 1, left and middle) is a mathematical sculpture designed by Koos Verhoeff. Its beams have an equilateral triangle as cross section and run in the directions of the space diagonals of a (tilted) cube. That makes *Swinger* a BCC (body-centered cubic) lattice path. Its name is a pun on *Swing*, a sculpture by Arie Berkulin (Fig. 2, left), constructed from square beams that follow the face diagonals of a cube (Fig. 2, right), making *Swing* a FCC (face-centered cubic) lattice path.

*Swing* is a Hamiltonian cycle on a regular tetrahedron. Thus, *Swing* is a regular constant-torsion polygon [5]: all beam segments have the same length, all angles between adjacent beam segments are the same (60°), and all dihedral angles between angle spanning planes of adjacent vertices are the same (70.53°). *Swinger* is not a Hamiltonian cycle on a regular tetrahedron, but it is a regular constant-torsion polygon with joint angles of 70.53° and dihedral angles of 60° (dual to *Swing*). Koos liked *Swinger* better than *Swing*, because the former, unlike the latter, has joints with some 'flush' beam faces: one beam face is coplanar across the joint.

## Obtetrahedrille

*Swinger* is a Hamiltonian cycle on a tetragonal disphenoid (a tetrahedron with congruent isosceles triangles as faces) of a special kind, viz. having two dihedral angles of 90° at the two longer edges, and four dihedral

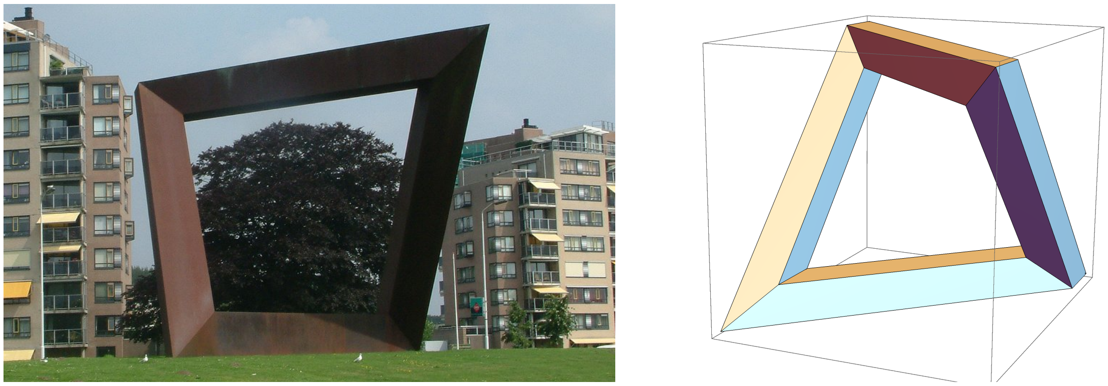

**Figure 2:** *Swing by Arie Berkulin, Cor-ten steel, 11 m high, Eindhoven, Netherlands, designed in 1969, constructed in 1977 (left; image source: [3]); computer model of Swing inside cube (right)*

angles of 60° at the four shorter edges (see Fig. 1, right, and Fig. 3, left, where the longer edges are aligned with the $Y$- and $Z$-axis, and the shorter edges are space diagonals of cubes). *Swinger* follows those shorter edges. Conway calls this shape an oblate tetrahedrille, or just *obtetrahedrille* [1, p.294], abbreviated here as OTHD. In [2], Gibb shows how to fold it from a single sheet of A4 paper. The OTHD can be embedded in space such that its vertices have integer coordinates, with two edges of length 2 and four edges of length $\sqrt{3}$. Unlike the regular tetrahedron, it is a space-filling polyhedron, resulting in what is known as the *tetragonal disphenoid honeycomb*. This is what makes the OTHD a good candidate for a modular building block.

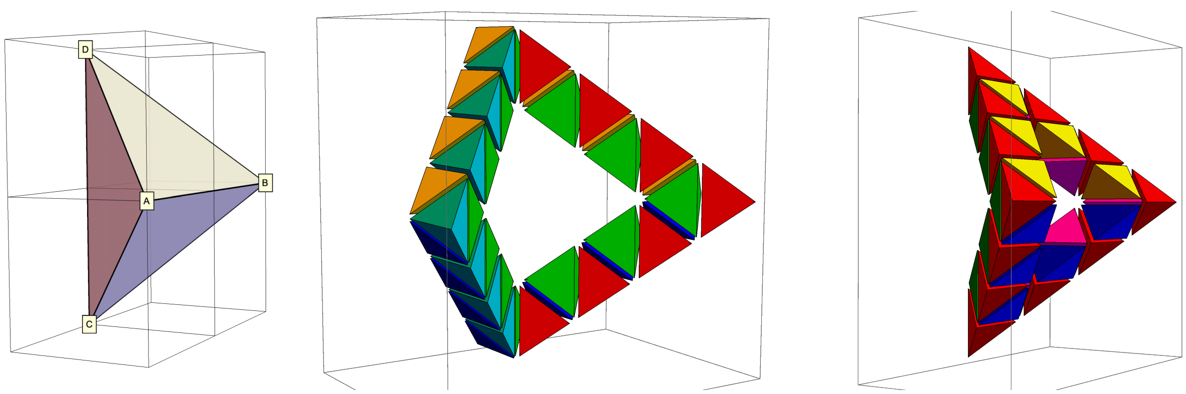


**Figure 3:** *Obtetrahedrille placed inside four cubes (left); Swinger (middle); variant of Swinger (right)*

What is even nicer is that the OTHD can be used to construct two types of beams:

- triangular beams, in the four directions of the cube's space diagonals (BCC lattice),
- square beams, in the three orthogonal directions (simple cubic, SC lattice).

Fig. 3 (middle) shows how *Swinger* is built up from OTHDs. On the right, a variant is shown where the cross section of the beams has been rotated through 180°. Fig. 4 shows various views of Kliekje ('Leftover'), constructed from square beams, being a regular constant-torsion polygon with bending (turtle turn) angles and dihedral (turtle roll) angles of 90°. Some other shapes constructed from OTHDs are shown in Fig. 5.

Note that these OTHD beams cannot make turns at arbitrary locations. Their branching points follow a helix around the beam. This is similar to the triangular beams obtained by folding $1 : \sqrt{2}$ rhombus strips described in [6]. The faces of the OTHD are half a $1 : \sqrt{2}$ rhombus. Thus, the OTHD can construct more varied shapes. For instance, *Swinger* cannot be constructed from $1 : \sqrt{2}$ rhombuses.

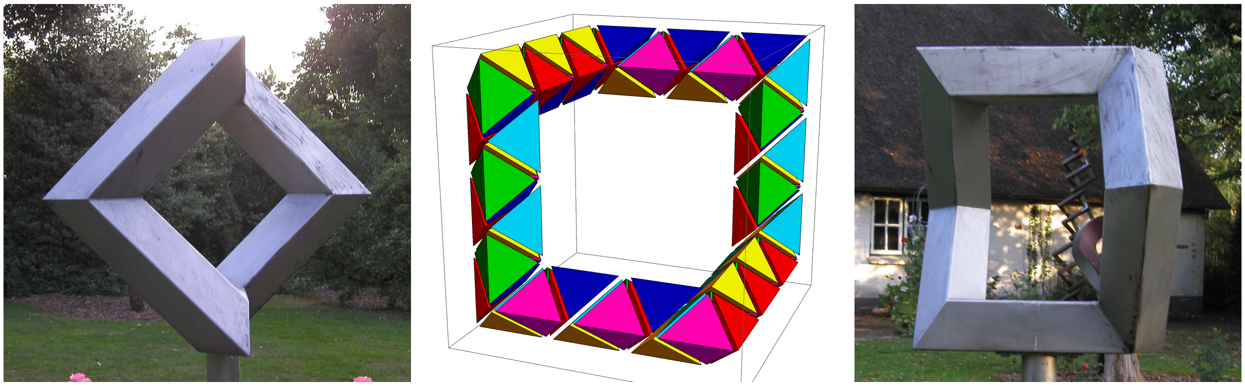

**Figure 4:** *Kliekje by Koos Verhoeff, stainless steel (various views, and its construction from OTHDs)*

**Figure 5:** *Trefoil and figure-8 knot from OTHDs (left, right); Rusty Thing by Koos Verhoeff, Corten (middle)*

## Language to Describe Obtetrahedrille Constructions

It is convenient to have a notation to describe OTHD constructions. We use a language based on 3D turtle geomertry [4], where the turtle supports these commands (Fig. 6, left):

- $B$ (Back) the face via which the turtle arrived at the OTHD
- $P$ (Perpendicular) place next OTHD on $P$-face
- $L$ (Left) place next OTHD on $L$-face
- $R$ (Right) place next OTHD on $R$-face
- $H_i$ (History) go back to state at step $i$ (negative for relative to last step)

The turtle travels in the lattice shown in Fig. 6 (middle). This is a sublattice of the FCC lattice, consisting of the edges of truncated octahedrons that fill space. The OTHDs are the Voronoi regions of this lattice.

A repetition of $R$ (or $L$) commands describes a triangular beam, where the turtle follows a period-3 helical path (Fig. 6, right a). A repetition of $PLH_{-1}R$ describes a square beam (Fig. 6, right b). A rhombic dodecahedron (Fig.7, left) is described by $LRLRRLRLLRLRRLRLLRLRRLRL = (ss')^3$ where $s = (LR)^2$ and $s' = (RL)^2$. The path traveled by the turtle is then a Hamiltonian cycle on the truncated octahedron.

It turns out that the OTHDs appear pairwise in three orientations, depending on the vector connecting the centers of their long edges. In Fig. 3, 4, and 5, the OTHD orientations are distinguished by six colors.

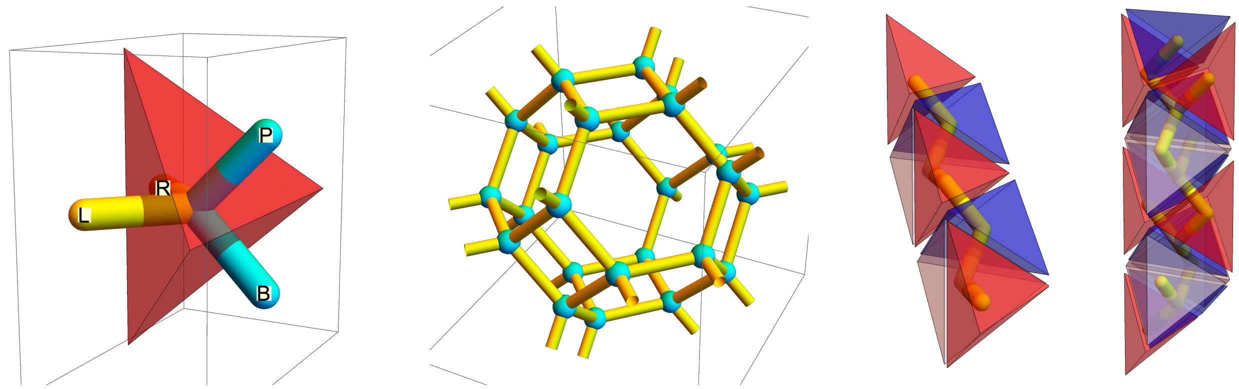


**Figure 6:** *Turtle commands (left), lattice (middle), triangular and square beams (right a, b)*

## Magnetizing the Obtetrahedrille

The faces of the OTHD can be magnetized with north and south poles, such that all modules are identical, and can be magnetically joined to fill space. Two faces sharing a longer edge get north poles, and the other two faces, sharing the other longer edge, get south poles. If Fig. 6 (left) the $P$ and $B$ faces have North poles, and the $L$ and $R$ faces have South poles. Fig. 7 (right) shows how everything connects properly.

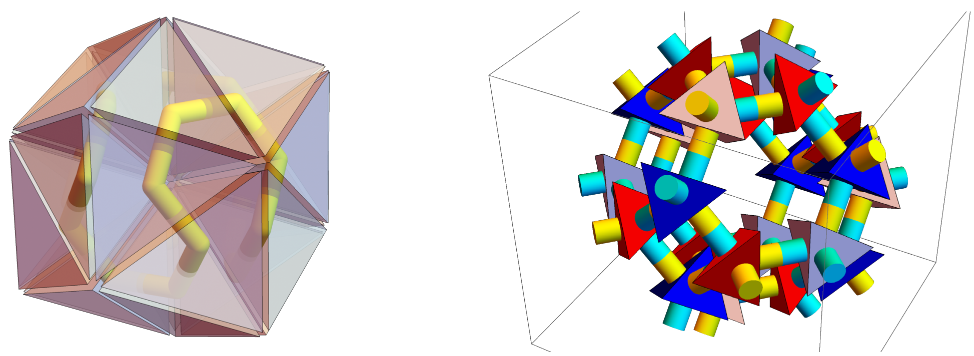

**Figure 7:** *Rhombic dodecahedron from 24 OTHDs (left); magnetized OTHDs (right)*

## References

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