Geared Jitterbugs

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Abstract

We describe geared versions of Buckminster Fuller's jitterbug mechanism, and a variant cuboctahedral mechanism. In the cuboctahedral case, differences between the triangular and square faces led us to a numerical method to construct acircular gears, allowing for rotation rates that vary with angle.

Introduction

In 1948, Buckminster Fuller discovered the jitterbug mechanism, a linkage consisting of eight rigid triangular faces joined at their corners by point hinges. The jitterbug expands from an octahedral to a cuboctahedral shape by twisting neighbouring triangles in opposite directions. See Figure 1 (a-c). However, this original version of the jitterbug has more than a single degree of freedom; other modes of movement are possible. In 1974, Dennis Dreher discovered that one could restrict the motion to only the expanding mode by replacing the point hinges between triangles with pairs of linear hinges attached to an intermediate part. See Figure 1 (d-f). Following Kiper and Söylemez [2], we refer to these intermediate parts as *DAP* parts, where "DAP" stands for "dihedral angle preserving", as they enforce the angle between the normals of adjacent triangular parts. See Schwabe [4] for a brief history of the jitterbug mechanism.

Our first contribution is to add bevel gear teeth onto the two triangular parts that mesh in the vicinity of a DAP part. Mathematically, the DAP parts reduce the possible motion of the jitterbug to a single degree of freedom. However, the real world does not behave: play in the hinges can result in the mechanism departing from this path to the extent that it can jam. The added gear teeth work in concert with the DAP parts to enforce preservation of the dihedral angles. Furthermore, the teeth constrain the relative rotation rates of the triangular parts. Both of these effects help to restrict the motion of the physical model to the desired single degree of freedom, resulting in a smoother mechanism.

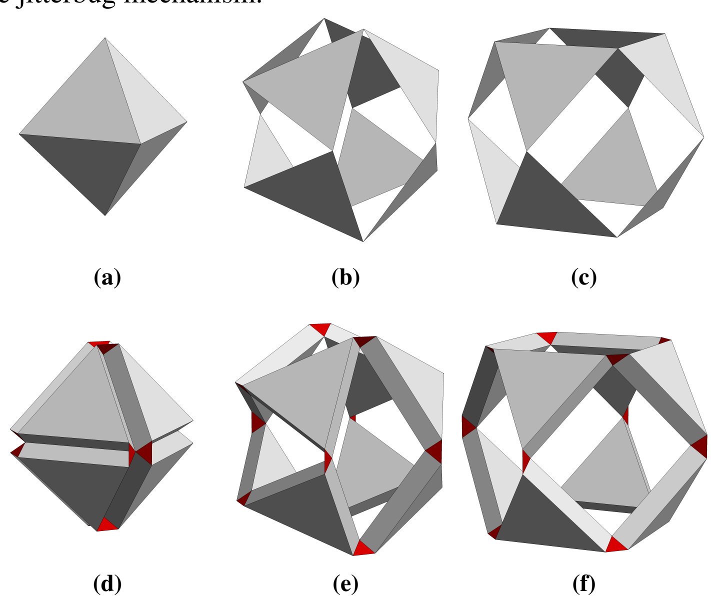


Figure 1: *(a-c) The jitterbug mechanism with point hinges. (d-f) The jitterbug mechanism with DAP parts.*

Verheyen [5], Kiper and Söylemez [2], and others have investigated generalizations of the jitterbug mechanism to other polyhedra. We apply our gearing technique to another example: a jitterbug mechanism based on a cuboctahedron which expands to form a rhombicuboctahedron. Here the gear design is complicated by the fact that the triangular and square faces of the mechanism must rotate at different, varying rates as the mechanism expands.

Gearing the Jitterbug

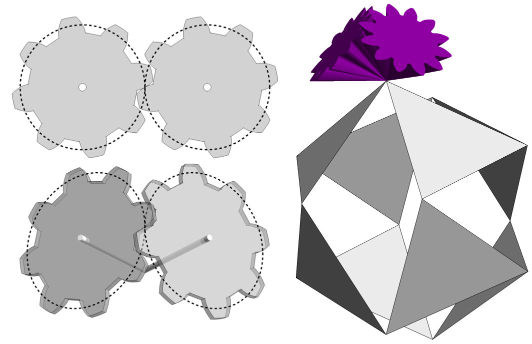


Figure 2: *Making conical gears.*

To design the bevel gears for our octahedral jitterbug mechanism, we used a two-dimensional profile for involute gears [1], coned to a point. Standard circular gears have parallel axes, while bevel gears have axes that intersect at a point. To construct a pair of bevel gears, take a pair of gears with parallel axes and rotate one of them about the common tangent vector to the two *midcurves* of the gears – curves of common tangency without teeth (here drawn dashed in Figure 2). The axes of the gears now meet in a single point. Coning the gear faces to this intersection point creates a set of bevel gears. The faces of the gears are always perpendicular to their axes, so this rotation angle determines the cone angles for both gears. The cone point for these bevel gears is the intersection of the two hinge axes on the DAP part, which is also a vertex of the underlying jitterbug mechanism with point hinges. The resulting geared mechanism is shown in Figure 3.

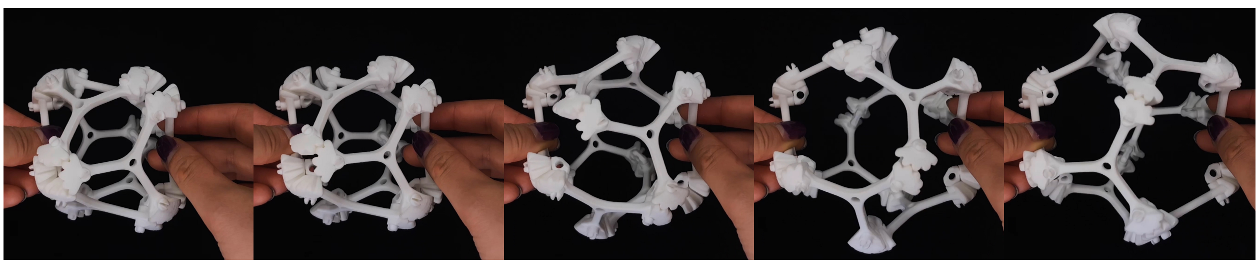


Figure 3: *3D printed geared jitterbug.*

Cuboctahedral Jitterbug

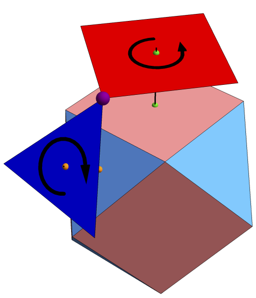


Figure 4

In the octahedral jitterbug, neighbouring triangles rotate in opposite direction. In the cuboctahedral jitterbug, the triangles all rotate in the same direction while the squares rotate in the other. When the mechanism expands from a cuboctahedron to a rhombicuboctahedron, the triangles rotate by $2\pi/3$, as they do in the octahedral design. However, the squares rotate by only $\pi/2$. Moreover, the angle of rotation of the triangles, θ say, is not linearly related to the angle of rotation of the squares, ϕ say. This makes gearing considerably more complicated than the octahedral case.

To see how to calculate the relationship between θ and ϕ, consider Figure 4. Here a red square and a blue triangle are joined at a point hinge and start as faces of a cuboctahedron. We rotate the triangle by an angle θ about an axis perpendicular to its face, while allowing it to also move outwards along that axis. We also allow the square to rotate and translate along its perpendicular axis. With these constraints, there is a line perpendicular to the triangle's axis on which the point hinge must lie. The point hinge must also lie on a cylinder centered on the square's axis. This line and cylinder intersect twice. One of these two intersections is the position of the hinge, and from this we derive the following relation between the rotation angles θ of the triangle and ϕ of the square:

$$\phi = \phi(\theta) = \arctan\left(\frac{-3\cos(\theta) + \sqrt{3}\left(\sin(\theta) + \sqrt{4 - \cos(2\theta) + \sqrt{3}\sin(\theta)}\right)}{3\cos(\theta) - \sqrt{3}\left(\sin(\theta) - \sqrt{4 - \cos(2\theta) + \sqrt{3}\sin(\theta)}\right)}\right) \tag{1}$$

Since the relation between θ and ϕ is not linear, the gears for this mechanism cannot be circular.

Acircular Gears

Consider two gears with distance one between their axles. Let the rotation angles of the driving and driven gears be θ and $\phi = \phi(\theta)$ respectively. Initially we think of these gears as being toothless but rolling against each other without slipping. The radii $a(\theta)$ and $b(\phi(\theta))$ thus add to one. Moreover, as the gears roll, no slipping means that the arc length traversed by the driving gear between angles 0 and θ is the same as the arc length traversed by the driven gear between angles 0 and ϕ, so

$$\int_0^\theta \sqrt{a(t)^2 + a'(t)^2}dt = \int_0^\phi \sqrt{b(t)^2 + b'(t)^2}dt.$$

For our cuboctahedron jitterbug, we have the relationship $\phi = \phi(\theta)$ given in equation (1), and we must solve an integral equation to find the radii $a(\theta)$ and $b(\phi)$ that will achieve this relationship. This is difficult to solve analytically, so we move to a numeric solution. We will use the set up shown in Figure 5.

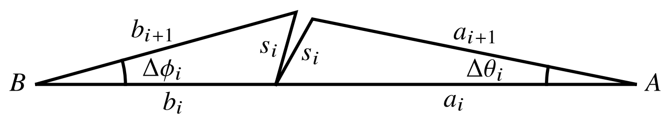


Figure 5

Here, the gear centered at A is the driving gear, and the gear centered at B is the driven gear. The driving gear has radius a_i when rotated by θ_i and the driven gear has complementary radius $b_i = 1 - a_i$. After the driving gear has been rotated by an additional $\Delta\theta_i$, it has radius a_{i+1}, and the additional arc length is s_i, where $s_i^2 = a_i^2 + a_{i+1}^2 - 2a_i a_{i+1}\cos(\Delta\theta_i)$ by the law of cosines. The no slipping condition means that the driven gear must also have arclength s_i, so by the law of cosines again, we get that $s_i^2 = b_i^2 + b_{i+1}^2 - 2b_i b_{i+1}\cos(\Delta\phi_i)$. We also know that the new radius of the driven gear must satisfy $b_{i+1} = 1 - a_{i+1}$. Putting all of this information together, we can solve:

$$a_{i+1} = \frac{(1 - a_i)(1 - \cos(\Delta\phi_i))}{1 - \cos(\Delta\phi_i) + a_i\cos(\Delta\phi_i) - a_i\cos(\Delta\theta_i)}$$

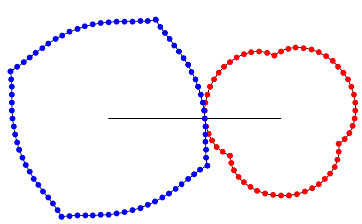


Figure 6: *Profiles of the triangular and square gear midcurves.*

Using the derivative $\phi'(\theta)$, we can work out the initial relative speeds of the gears, and therefore the initial value a_0. We chose a constant driving step size $\Delta\theta_i = \delta = \pi/300$. From this we calculate the driven step size $\Delta\phi_i = \phi((i+1)\delta) - \phi(i\delta)$. We can then generate all of the radii a_i and b_i. The gear profiles we obtain by applying this technique with equation (1) are shown in Figure 6.

Note that this numerical method does not suffer much from "drift": so long as the gears we generate keep matching up at the corners of subsequent triangles (as in Figure 5), then the relative rotation rates will be correct.

Adding Teeth

The midcurve of the acircular gears prescribe the required rotation rate of the square gear with respect to the triangular gear due to the no-slip condition. However, as these just barely touch, there would be nothing except friction to transfer torque from the triangular gear to the square gear to induce rotation. Circular gears use involute teeth to more efficiently transfer torque. To add teeth to our acircular gears, we carve out teeth using a linear rack (black in Figure 7) with the same total length as the arc length of the midcurve of the acircular gears. This is similar to a method described by Laczic [3]. As the gears rotate, the midcurve of the rack is oriented to be tangent to the midcurves of the acircular gears at the point of common tangency. The rack also rolls without slipping along the midcurve of the acircular gears so that as the triangular gear rolls through an angle, all three midcurves will have moved by the same arc length. The tooth profile is the envelope of the rack profile as this motion is executed. See Figure 8. The benefit of this design is that the pressure angle of the teeth can be set easily by the linear rack. This ensures that a constant torque can be transferred between gears.

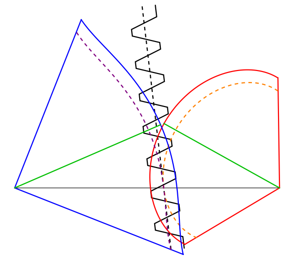
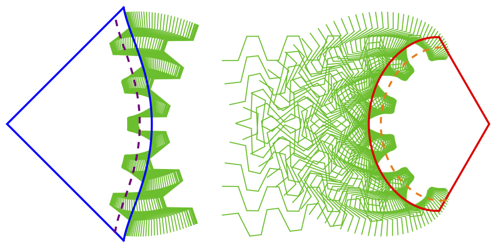
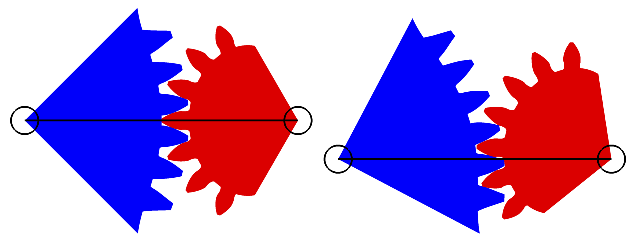


Figure 7: *Cutting teeth.*



(a) *Gear teeth cut by the envelope of a rack profile.*



(b) *Gear tooth profile.*

Figure 8

Result

The final kinetic sculpture (Figure 9) consists of three type of parts: square faced parts (×6) which sit at the square faces of the cuboctahedron, triangular faced parts (×8) which sit at the triangular faces of the cuboctahedron, and connectors (×24) which align each pair of triangular/square acircular bevel gears.

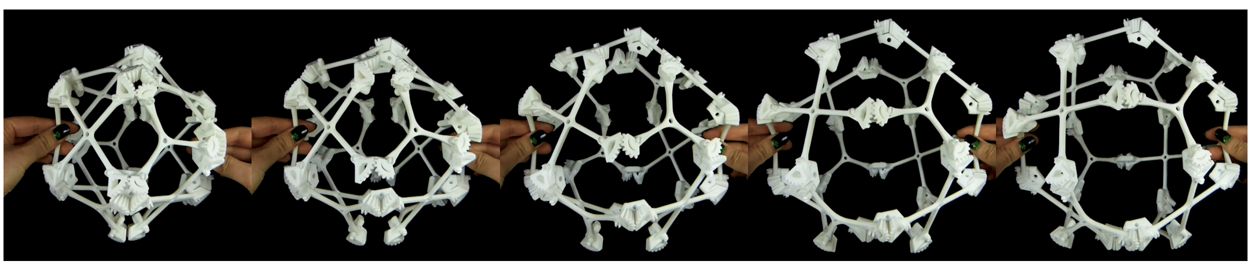


Figure 9: *3D printed geared cuboctahedral jitterbug.*

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